IE201 CLASS NOTES

A smooth sea never made a skillful sailor.
English Proverb

A great teacher is one who realizes that he himself is also a student and whose goal is not dictate the answers, but to stimulate his students creativity enough so that they go out and find the answers themselves.

Herbie Hancock

If you are not willing to learn, no one can help you.
If you are determined to learn, no one can stop you.
Zig Ziglar

ACKNOWLEDGMENTS

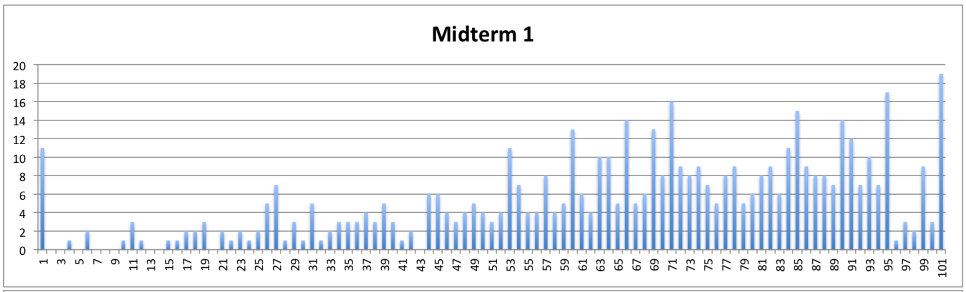
I would like to thank Meserret Karaca for typing answers for the questions provided here, and helping me prepare this LaTeX document.

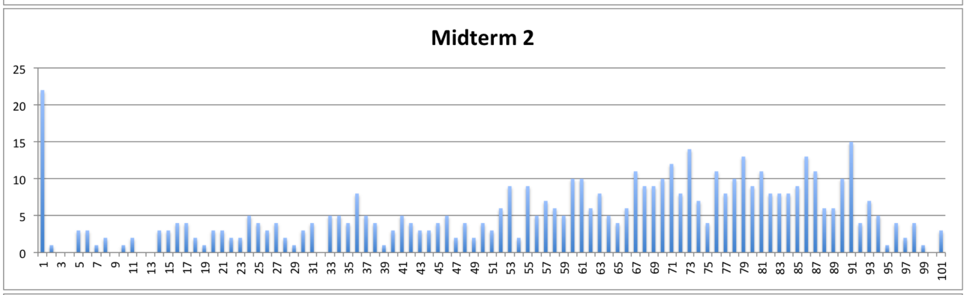
Table of Contents

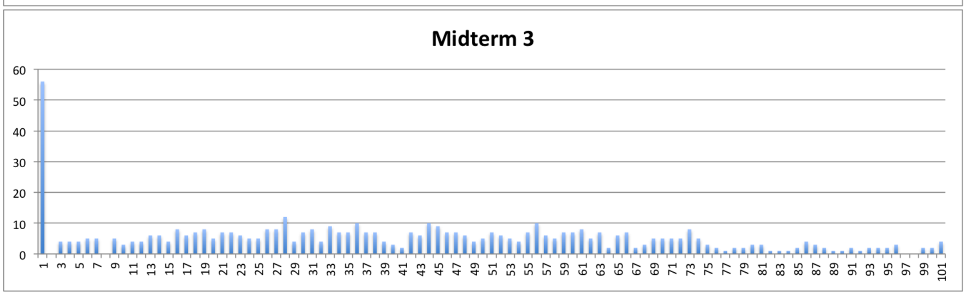
Course Statistics

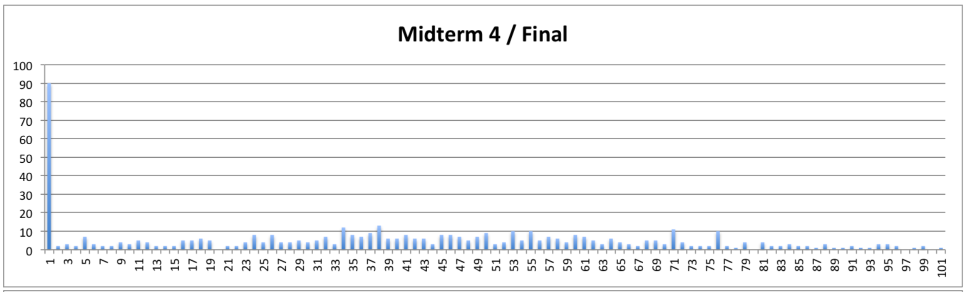
This section presents some statistics regarding the 531 students who have taken this course until the beginning of Spring 2016-2017 semester. 288 of these students passed the course. Here are some stats regarding the exams.

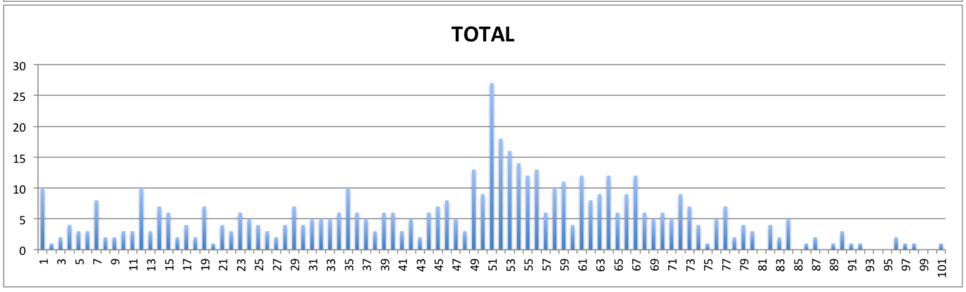
All Students











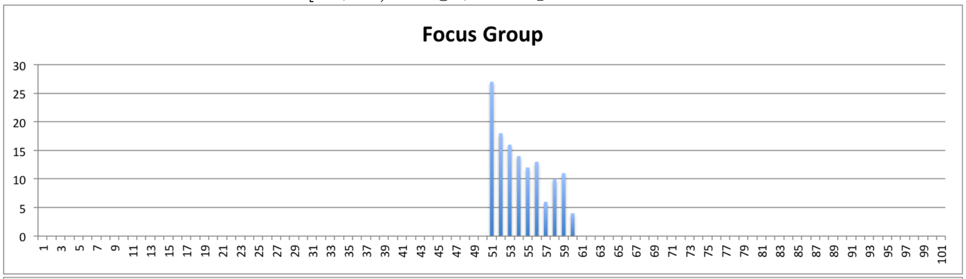
Note that the peak right above 50.00 line is coincidental and certainly an outcome of students' effort. I never look at the names or previous exam grades of students when grading papers. Below is another proof of that:

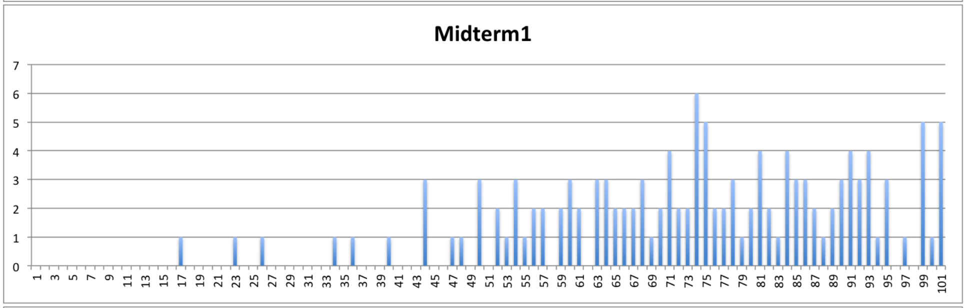
50.15	D
50.1	D
50.1	D
50.025	D
50	D
50	D
50	D
49.6	F
49.55	F
49.55	F
49.5	F
49.175	F
49.05	F
49.05	F
49	F
49	F
48.9	F
48.75	F
48.65	F
48.6	F
48.5	F
48.45	F
48.425	F
48.4	F

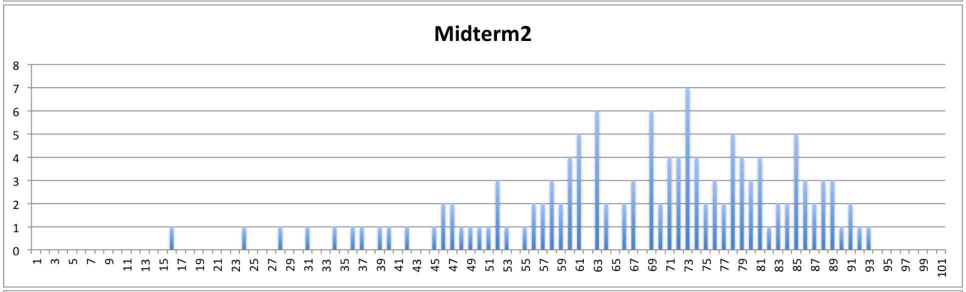
Next, I present stats for a focus group of 131 students, who barely passed the course.

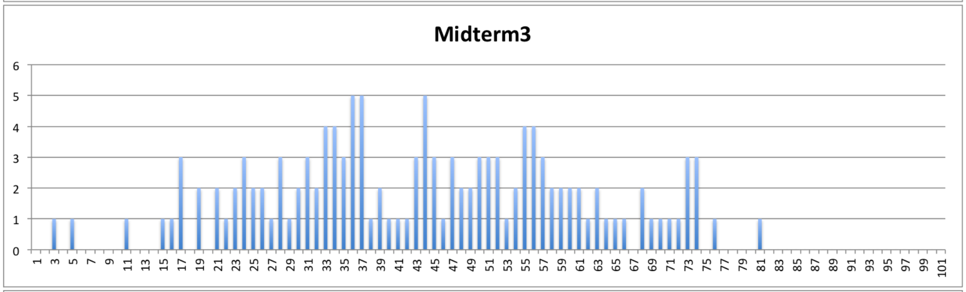
Students who Barely Passed

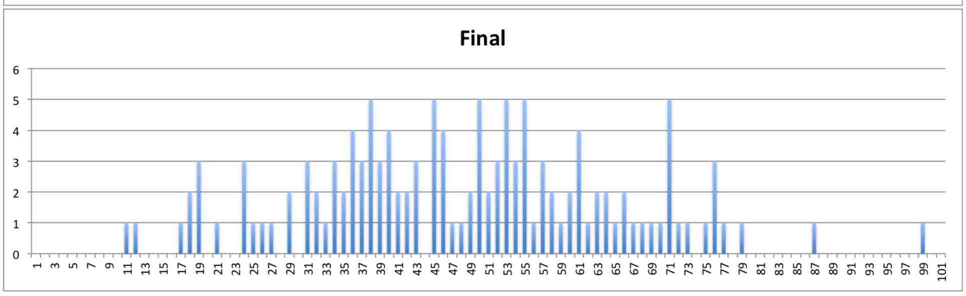
These students are in the $[50, 60)$ range, thus passed with either D or D+.











Some conclusions I derive from this analysis:

- In Midterm 1, you should receive at least 60, preferably 80's-90's (which is doable as shown above)
- In Midterm 2, you should stay in the upper semi-half of this normal distribution of mean 70. This is the exam with most mechanical work; minimal creativity needed. Therefore, *the higher you get, the easier it will be later.*
- You need to work really hard in class and during office hours after Midterm 2. Deep understanding, more insights, and some creativity needed. Exam 3 will actually be the final exam for those who are shooting for *just passing the course.* Half of the focus group received more than 43 in Midterm 3. Let me put it this way; a minimum of 40 points in Midterm 3 is a lot more than $40 \times 30\%=12$ points in your course total; it ensures that you know the fundamentals of the course and you can do good in the final.
- Despite being comprehensive, final exam stats is similar to Midterm 3, which supports my claim above. Some semesters Final is slightly harder, some semesters Midterm 3. You should not be disconnected from the subject matter, certainly receive something above 30's in the Final.
- Those who passed *consistently* receive above 30 in all exams. A couple of exceptions *definitely* make it up in another (probably earlier) exam by showing an extraordinary effort.

Bottomline

Never get below 20 in any exam! Collect a total of at least 150 points in the first 2 Midterm exams. Remember, **the real challenge starts after Midterm 2**! It is very unlikely to make-up a bad Midterm 3 grade in the Final.

Chapter 1

Mathematical Modeling in Operations Research (Weeks 1-3)

1.1 Optimization

Optimization is the most vital aspect of Operations Research (OR). Mathematical Modeling is the heart of optimization. So we start with the details, requirements, and examples of optimization models[1].

The business environment today obviously poses unique and unprecedented opportunities and challenges. The unexpectedly rapid fall of global trade barriers and the increasing fluidity of information have created a cut-throat, competitive business environment. In reaction, new business platforms and tools have surfaced to deal with the challenges of this flood of data.

Twenty years ago, few "civilians" had heard of web marketing analytics, supply chains, or revenue management. But with the rise of Google and Wal-Mart and low fare airlines, these terms have become an essential part of the conversation. What do they have in common? All use data in large volumes to drive profitable decisions. And in many of them, optimization is the "secret sauce."

Optimization is simply a mathematically sophisticated way to represent a business process in software. Built well, an optimization model uses relevant data and business rules to recommend decisions that generate the best possible result. The three key elements of an optimization model are

(1) Objectives: These are your business goals. Objectives can include goals such as maximizing profit, shrinking the quote-to-cash cycles, reducing shipment costs, or whatever else you want to minimize or maximize.

(2) Decision Variables: These are geek-speak for decisions in your control. Suppose you manage a refinery. Is the price of crude oil in your control? No, no matter how much you might wish it were. How about how much a certain type of crude you refine? Yes.

[1] This subsection and next are from the booklet titled "Optimization for Dummies" by Sanjay Saigal.

In your production model, the amount of crude oil will be a variable. The forecast oil price? Simply data (or parameter).

(3) <u>Constraints</u>: Think of constraints as business rules or realities relevant to the optimization model. A factory only has so many workers on a shift. A portfolio manager is not allowed to invest more than a certain fraction in the tech sector. An arriving aircraft requires so many minutes for cleaning and refueling. All of these are constraints.

Put these three types of elements together and you get an *optimization model.* Program the optimization model into a computer and integrate it with a *solver* (a king of number crunching machine) and a user interface, and you have yourself an *optimization system.* If fed good data, the system outputs *best possible* decisions.

1.2 When Should You Optimize?

The first characteristic of an "optimizable" process is rich data. Optimization perfectly illustrates the adage "garbage in, garbage out." The more accurately you measure what you're doing, the greater your potential to improve.

As an example, delivery services such as FedEx and UPS minimize operation costs in large part through optimization. As part of their rich data collection, they track turn-by-turn locations of each vehicle, real-time traffic, operating costs, and continuously refreshed pick-up and drop-off information.

Another key factor for applying optimization is repeatability. While optimization has been applied with success to one-off decisions, the U.S. Federal Communications Commission's 2008 auction of cellphone frequencies comes to it is more typically used to improve decisions that occur regularly, even frequently. For example, the National Broadcasting Company (NBC) uses optimization to sell "up front" television advertising every year, Marriott and Intercontinental use it to optimize their room inventory, and Hewlett Packard uses it to plan and run manufacturing. These are all repetitive processes where squeezing out even a 0.1 percent gain in efficiency can translate to millions of dollars.

Finally, optimization is most suited for complex, resource-constrained situations. Advertising seconds are a television network's most precious commodity. The same holds true of rooms in a hotel for a hotel chain or expensive machine and staff time on a factory floor.

While reading this chapter, you might be thinking, "This sounds good, but aren't my IT systems already doing all this? Isn't that what I should expect from my company's Customer Relationship Management (CRM) or Enterprise Resource Planning (ERP) or Business Process Management (BPM) system?" No. Not really.

Most business IT systems — such as CRM and ERP — are essentially data repositories with a light sprinkling of rule-based automation. For example, the Salesforce Automation capability in your CRM may assign new leads to salespersons. That's decision-making of a sort. But such assignment rules have to be defined up front, such as: "Calls from Canada are routed to Tracy." The CRM can't tell you which salesperson should be assigned the lead to maximize profitability; an optimization-based system can.

Typically, the most challenging part of mathematical modeling in OR (especially for a Linear Program (LP)) is defining *decision variables*.

⋆ What do you understand from the term Linear Program?

Constraints and objective function must be linear functions of decision variables & decision variables must be continuous variables.

⋆ What is a linear function?

A linear function is a polynomial function of degree zero or one. e.g., $f(y_1, y_2, y_3) = 3y_1 + 5y_2 + 6, f(x_1, x_2) = 3x_1 + 5x_2, f(x_1) = 5$ *etc.*

⋆ What is a continuous variable?

A variable in real space with no imposition of additional constraints (except possible upper and lower bounds). e.g., $x_1 \geq 0, -3 \leq x_2 \leq 5$ *etc.*

1.3 LP Models

Example 1.1

The Metalco Company desires to blend 40 lbs of a new alloy (ALLOYA) from several available alloys having the following properties:

	Alloys				
Property	1	2	3	4	5
Percentage of tin	60	25	45	20	50
Percentage of zinc	10	15	45	50	40
Percentage of lead	30	60	10	30	10
Cost ($/lb)	22	20	25	24	27

Note that the new blend might consist of a single type or multiple types of alloy. The objective is to determine the proportions of these alloys that should be blended to produce the new alloy at a minimum cost. Formulate a linear programming model for this problem.

Suppose ALLOYA should contain at least 35 percent tin, 30 percent zinc, and 20 percent lead. How would you update your linear programming model?

<u>Decision variables:</u>

x_i: Amount of alloy i to be used in new alloy, $i = 1, 2, 3, 4, 5$

<u>Model:</u>

$$
\begin{array}{ll}
\min & z = 22x_1 + 20x_2 + 25x_3 + 24x_4 + 27x_5 \\
\text{s.t.} & .60x_1 + .25x_2 + .45x_3 + .20x_4 + .50x_5 \geq .35 \times 40 \\
& .10x_1 + .15x_2 + .45x_3 + .50x_4 + .40x_5 \geq .30 \times 40 \\
& .30x_1 + .60x_2 + .10x_3 + .30x_4 + .10x_5 \geq .20 \times 40 \\
& x_1 + x_2 + x_3 + x_4 + x_5 = 40 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

Suppose ALLOYA desires to blend at least 40 lbs (instead of precisely 40 lbs). How would you update your linear programming model?

Example 1.2 Coalco produces coal at three mines and ships it to four customers. The cost per ton of producing coal, the ash and sulfur content (per ton) of coal, and production capacity (in tons) for each mine are given in the first table below. The minimum amount of coal demanded by each customer are given in the second table.

The cost (in dollars) of shipping a ton of coal from a mine to each customer is given in the third table below. It is required that the total amount of coal shipped contain at most 5% ash and at most 4% sulfur. Formulate an LP that minimizes the cost of meeting customer demand.

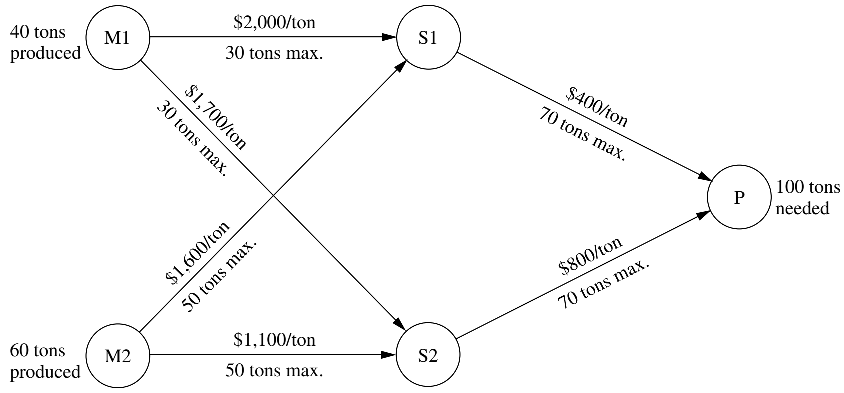
Mine	Prod. Cost ($)	Capacity	Ash Content (tons)	Sulfur Content (tons)
1	50	120	.08	.05
2	55	100	.06	.04
3	62	140	.04	.03

Customer 1	Customer 2	Customer 3	Customer 4
80	70	60	40

Mine	Customer 1	Customer 2	Customer 3	Customer 4
1	4	6	8	12
2	9	6	7	11
3	8	12	3	5

If, upon departure from **the third mine**, the company cheats by including sand into coal and doubling the weight of all trucks to all customers, how would you update your formulation? Assume the customers won't notice so that would be OK to satisfy the demand, however, that should inevitably affect ash and sulfur percentages.

Example 1.3 The Fagersta Steelworks currently is working two mines to obtain its iron ore. This iron ore is shipped to either of two storage facilities. When needed, it then is shipped on to the company's steel plant. The diagram below depicts this distribution network, where M1 and M2 are the two mines, S1 and S2 are the two storage facilities, and P is the steel plant. The diagram also shows the monthly amounts produced at the mines and needed at the plant, as well as the shipping cost and the maximum amount that can be shipped per month through each shipping lane. Management now wants to determine the most economical plan for shipping the iron ore from the mines through the distribution network to the steel plant. Formulate a linear programming model for this problem.



If the production in M1 increases to 50 tons and S2 demands 10 tons, an arc from S1 to S2 (capacity 20 tons, cost $1000/ton), from M1 to M2 (capacity 20 tons, cost $400/ton) and from P to M2 (capacity 10 tons, cost $100/ton) are added, how can you formulate an LP?

Example 1.4 The MJK Manufacturing Company must produce two products in sufficient quantity to meet contracted sales in each of the next three months. The two products share the same production facilities, and each unit of both products requires the same amount of production capacity. The available production and storage facilities are changing month by month, so the production capacities, unit production costs, and unit storage costs vary by month. Therefore, it may be worthwhile to overproduce one or both products in some months and store them until needed.

For each of the three months, the second column of the following table gives the maximum number of units of the two products combined that can be produced on Regular Time (RT) and on Overtime (OT). For each of the two products, the subsequent columns give (1) the number of units needed for the contracted sales, (2) the cost (in thousands of dollars) per unit produced on Regular Time, (3) the cost (in thousands of dollars) per unit produced on Overtime, and (4) the cost (in thousands of dollars) of storing each extra unit that is held over into the next month. In each case, the numbers for the two products are separated by a slash /, with the number for Product 1 on the left and the number for Product 2 on the right.

	Maximum Combined Production		Product 1/Product 2			
				Unit Cost of Production ($1,000's)		Unit Cost of Storage
Month	**RT**	**OT**	**Sales**	**RT**	**OT**	**($1,000's)**
1	10	3	5/3	15/16	18/20	1/2
2	8	2	3/5	17/15	20/18	2/1
3	10	3	4/4	19/17	22/22	

The production manager wants a schedule developed for the number of units of each of the two products to be produced on Regular Time and (if Regular Time production capacity is used up) on Overtime in each of the three months. The objective is to minimize the total of the production and storage costs while meeting the contracted sales for each month. There is no initial inventory, and no final inventory is desired after the three months.

Formulate the problem as a linear program.

Decision variables:

PRT_{ij} : Product j produced at month i in regular time $(i = 1, 2, 3, j = 1, 2)$

POT_{ij} : Product j produced at month i in over time $(i = 1, 2, 3, j = 1, 2)$

I_{ij} : Inventory of product j after month i $(i = 1, 2, 3, j = 1, 2)$

Model:

$\min PRT_{11} \times 15 + PRT_{12} \times 16 + PRT_{21} \times 17 + PRT_{22} \times 15 + PRT_{31} \times 19 + PRT_{32} \times 17 + POT_{11} \times 18 + POT_{12} \times 20 + POT_{21} \times 20 + POT_{22} \times 18 + POT_{31} \times 22 + POT_{32} \times 22 + I_{11} \times 1 + I_{12} \times 2 + I_{21} \times 2 + I_{22} \times 1$

subject to

$PRT_{31} + POT_{31} + I_{21} = 4$

$PRT_{32} + POT_{32} + I_{22} = 4$

$I_{11} = PRT_{11} + POT_{11} - 5$

$I_{12} = PRT_{12} + POT_{12} - 3$

$I_{21} = PRT_{21} + POT_{21} + I_{11} - 3$

$I_{22} = PRT_{22} + POT_{22} + I_{12} - 5$

$PRT_{11} + PRT_{12} \leq 10$

$POT_{11} + POT_{12} \leq 3$

$PRT_{21} + PRT_{22} \leq 8$

$POT_{21} + POT_{22} \leq 2$

$PRT_{31} + PRT_{32} \leq 10$

$POT_{31} + POT_{32} \leq 3$

$PRT_{ij}, POT_{ij} \geq 0 \quad (i, j = 1, 2)$

$I_{ij} \geq 0 \quad (i = 1, 2, 3, j = 1, 2)$

Example 1.5 (Shipbuilding Company) A shipbuilding company is planning its workforce and inventory over the upcoming summer. The company can produce in a month and satisfy the same month's demand. It is expected that there will be 400 workers on payroll and no inventory in the beginning. Building a ship requires 275 workers. Monthly salary for each worker is 2,000 TL. Cost of hiring and firing a worker are 400 TL and 1,000 TL, respectively. Inventory cost per ship per month is 5,000 TL. Expected demands for June, July and August are 1, 4, and 2 and the demand must definitely be satisfied with no delays. Production costs in these three months are 20,000 TL, 40,000 TL and 30,000 TL, respectively. Model the problem to minimize the cost.

Solution

Decision Variables:

W_t: Number of workers in month t, $t = 0, 1, 2, 3$, $t = 1$ implies June...

P_t: Number of ships produced in month t, $t = 1, 2, 3$.

I_t: Number of ships on inventory at the end of month t, $t = 0, 1, 2, 3$.

H_t: Number of workers hired at t^{th} month, $t = 1, 2, 3$.

F_t: Number of workers fired at t^{th} month, $t = 1, 2, 3$.

Objective Function:

min $\quad 2,000\sum_{t=1}^{3} W_t + 20,000P_1 + 40,000P_2 + 30,000P_3 + 400\sum_{t=1}^{3} H_t + 1,000\sum_{t=1}^{3} F_t + 5,000\sum_{t=1}^{3} I_t$

Constraints:

$W_t \geq 275P_t, \quad t = 1, 2, 3$

$W_{t-1} + H_t - F_t = W_t, \quad t = 1, 2, 3$

$W_0 = 400$

$I_{t-1} + P_t - D_t = I_t, \quad t = 1, 2, 3$

$I_0 = 0$ (that makes the nonnegativity of I_0 redundant)

$W_t, P_t, I_t, H_t, F_t \geq 0, \quad t = 1, 2, 3$

Note: D_t is a parameter in this problem, where $D_1 = 2, D_2 = 4, D_3 = 1$.

Example 1.6 (Shipbuilding Company with Backordering) A shipbuilding company is planning its workforce and inventory over the upcoming summer. The company can produce in a month and satisfy the same month's demand. It is expected that there will be 400 workers on payroll and no inventory in the beginning. Building a ship requires 275 workers. Monthly salary for each worker is 2,000 TL. Cost of hiring and firing a worker are 400 TL and 1,000 TL, respectively. Inventory cost and backordering cost per ship per month are 5,000 and 40,000 TL. Expected demands for June, July and August are 1, 4, and 2 and production costs in these months 20,000 TL, 40,000 TL and 30,000 TL, respectively. Model the problem to minimize the cost.

Solution

Decision Variables:

W_t: Number of workers in month t, $t = 0, 1, 2, 3$, $t = 1$ implies June...

P_t: Number of ships produced in month t, $t = 1, 2, 3$.

B_t: Number of ships backordered in month t, $t = 0, 1, 2, 3$.

I_t: Number of ships on inventory at the end of month t, $t = 0, 1, 2, 3$.

H_t: Number of workers hired at t^{th} month, $t = 1, 2, 3$.

F_t: Number of workers fired at t^{th} month, $t = 1, 2, 3$.

Objective Function:

min $\quad 2,000\sum_{t=1}^{3} W_t + 20,000P_1 + 40,000P_2 + 30,000P_3 + 400\sum_{t=1}^{3} H_t + 1,000\sum_{t=1}^{3} F_t + 40,000\sum_{t=1}^{3} B_t + 5,000\sum_{t=1}^{3} I_t$

Constraints:

$W_t \geq 275P_t, \quad t = 1, 2, 3$

$W_{t-1} + H_t - F_t = W_t, \quad t = 1, 2, 3$

$W_0 = 400$

$I_{t-1} - B_{t-1} + P_t - D_t = I_t - B_t, \quad t = 1, 2, 3$

$I_0 = 0$, $B_0 = 0$ (that makes the nonnegativity of I_0 and B_0 redundant)

$W_t, P_t, B_t, I_t, H_t, F_t \geq 0, \quad t = 1, 2, 3$

$\star$ If the company owed 10,000 TL independent from hiring / firing / production etc, would you change your answer? No, that would have only *offset* the objective function.

$\star$ If the costs (objective function coefficients) were 20, 200, 400, 300, 4, 10, 400, 50TL, respectively, would you change your answer? No, that would have only *scaled* the objective function.

$\star\star\star$ Suppose the demand for summer has to be met at the end of August (although there might be backorders in June or July). How would you update your formulation? Add constraint $B_3 = 0$ only.

NOTES

1. A *feasible* solution satisfies "all" constraints.
2. An *infeasible* solution violates "at least one" constraint.
3. An *optimal* solution is a *feasible* solution that provides "the most favorable" objective function value. (The largest for maximization problems and the smallest for minimization problems)
4. A problem might have multiple optimal solutions.
5. Best **corner point feasible (CPF) solution** must be an *optimal* solution.
6. Constraints are typically categorized as *functional* constraints and *nonnegativity* constraints.
7. A problem might have no optimal solution in the following 2 cases:
   [7.1] Infeasibility.
   [7.2] Unboundedness.
8. From now on, instead of labeling the objective function and constraints explicitly, we will denote any optimization problem as follows: max / min [*objective function*]
   s.t. [*constraints*]
   where s.t. stands for subject to.
9. Never use strict inequalities in an LP formulation. Only $\leq$ and $\geq$ signs have to be used.
10. In the world of OR, we start with LP, but we use IP, NLP, MINLP, QP, SDP, BLP exclusively.

1.4 Exercises

(1) How many feasible solutions can an LP problem have? 0, 1, 2, infinitely many?

(2) How many decision variables are used in Example 1.3? Is it possible to formulate the same question with a different number of decision variables?

(3) How many functional constraints and decision variables are there in Example 1.6?

(4) Find two feasible and two infeasible solutions for the problem in Example 1.3.

(5) Try and solve **Reclaiming Solid Wastes** example in Section 3.4 of your textbook on your own. If you cannot, make sure that you read and understand the solution in the book.

(6) A company produces two different products by mixing two different raw materials. Each kg of product 1 can be sold for 10 TL and each kg of product 2 can be sold for 8 TL. The processing cost per kg of raw material is 2 TL. Raw material 1 costs 1 TL/kg and raw material 2 costs 1.5 TL/kg. The demand for product 1 is 200 kg and demand for

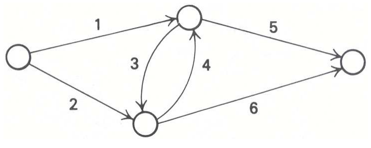
product 2 is 400 kg. Due to the agreements with the suppliers you need to purchase 150 kg of each type of raw material. The sulfur content of raw material 1 is 3% and raw material 2 is 2%. It is required that the sulfur content of product 1 is at most 2.2% sulfur and the sulfur content of product 2 is at most 2.5%. The usage of any type of raw material cannot exceed 75% of total usage. Formulate a linear programming problem to minimize the total cost. Formulate another linear programming problem to maximize the total profit.

(7) XComputers makes quarterly decisions about their product mix. They produce notebook computers, desktop computers and tablet. There are a number of limits on what XComputers can produce. The major constraints are as follows:

- Each computer (notebook, desktop or tablet) requires a Processing Chip. Due to tightness in the market, our supplier has allocated a total of 15,000 chips to us.
- Each computer requires memory. Memory comes in 1GB chip sets. A notebook computer has 2GB, a desktop computer has 4GB, and a tablet has 1GB memory installed. We received a great deal on chip sets, so have a stock of 25,000 chip sets (1GB each) to use over the next quarter.
- Each computer requires assembly time. Due to tight tolerances, a notebook computer takes 5 minutes to assemble while assembling a desktop takes 4 minutes and a tablet 3 minutes. There are 25,000 minutes of assembly time available in the next quarter.
- Given current market conditions, material cost, and our production system, each notebook computer produced generates \$600 profit, each desktop produces \$700 profit and each tablet produces \$500 profit.

Formulate a Linear Programming model for this problem to maximize the profit for XComputers.

(8) A gas company owns a pipeline network, sections of which are used to pump natural gas from its main field to its distribution center. The network is shown below, where the direction of the arrows indicates the only direction in which the gas can be pumped. The pipeline links of the system are numbered one through six, and the intermediate nodes are large pumping stations.



At the present time, the company nets 1200 mcf (million cubic feet) of gas per month from its main field and must transport that entire amount to the distribution center. The following are the maximum usage rates and costs associated with each link:

	1	2	3	4	5	6
Maximum usage: mcf/month	500	900	700	400	600	1000
Tariff: \$/mcf	20	25	10	15	20	40

The gas company wants to find those usage rates that minimize total cost of transportation. Formulate the problem as a linear program.

(9) The Primo Insurance Company is introducing two new product lines: special risk insurance and mortgages. The expected profit is \$5 per unit on special risk insurance and \$2 per unit on mortgages. Management wishes to establish sales quotas for the new product lines to maximize total expected profit. The work requirements are as follows: Formulate a linear programming model for this problem.

	Work-Hours per Unit		
Department	Special Risk	Mortgage	Work Hours Available
Underwriting	3	2	2400
Administration	0	1	800
Claims	2	0	1200

(10) Weenies and Buns is a food processing plant which manufactures hot dogs and hot dog buns. They grind their own flour for the hot dog buns at a maximum rate of 200 pounds per week. Each hot dog bun requires 0.1 pound of flour. They currently have a contract with Pigland, Inc., which specifies that a delivery of 800 pounds of pork product is delivered every Monday. Each hot dog requires $\frac{1}{4}$ pound of pork product. All the other ingredients in the hot dogs and hot dog buns are in plentiful supply. Finally, the labor force at Weenies and Buns consists of 5 employees working full time (40 hours per week each). Each hot dog requires 3 minutes of labor, and each hot dog bun requires 2 minutes of labor. Each hot dog yields a profit of \$0.20, and each bun yields a profit of \$0.10.

Weenies and Buns would like to know how many hot dogs and how many hot dog buns they should produce each week so as to achieve the highest possible profit. Formulate a linear programming model for this problem.

(11) Web Mercantile sells many household products through an on-line catalog. The company needs substantial warehouse space for storing its goods. Plans are being made for leasing warehouse storage space over the next 5 months. Just how much space will be required in each of these months is known. However, since these space requirements are quite

different, it may be most economical to lease only the amount needed each month on a month-by-month basis. On the other hand, the additional cost for leasing space for additional months is much less than leasing for one month only, so it may be less expensive to lease the maximum amount needed for the entire 5 months. Another option is the intermediate approach of changing the total amount of space leased (by adding a new lease and/or having an old lease expire) at least once but not every month. The space requirement and the leasing costs for the various leasing periods are as follows:

Month	Required Space
1	30,000 sq. ft.
2	20,000 sq. ft.
3	40,000 sq. ft.
4	10,000 sq. ft.
5	50,000 sq. ft.

Leasing Period	Cost per sq. ft.
1 month	\$65
2 months	\$100
3 months	\$135
4 months	\$160
5 months	\$190

The objective is to minimize the total leasing cost for meeting the space requirements. Formulate a linear programming model for this problem.

(12) Consider a restaurant that is open seven days a week. Based on past experience, the minimum number of workers needed on a particular day is given as follows:

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Number of workers needed	14	13	15	16	19	18	11

Every worker works five consecutive days, and then takes two days off, repeating this pattern indefinitely. Formulate the linear programing model that would minimize the number of workers that work for the restaurant.

(13) Suppose an investor has \$100 on Monday. Every day of the week (Monday through Friday), the investor has the following investment opportunity available: if s/he invests x dollars on a day and matches that initial investment (invests the same amount) the next day, then s/he will receive a total return of $3x$ dollars on the third day. That is, with a total investment of $2x$ dollars over two days, the investor receives $3x$ dollars on the third day. The investor wishes to determine an investment schedule that maximizes his total cash by the end of Saturday. Note that the investor cannot invest on Saturday

but s/he can receive a return on Saturday. Formulate a linear programming model to solve this problem.

1.5 Further Reading and Exercises

The reader is refereed to Chapters 2 and 3 in the textbook, *Introduction to Operations Research* by Hillier and Lieberman.

Chapter 2

Solving a Linear Program (Weeks 4-7)

Important Note: Throughout this course, we only work with continuous variables. In most of the questions, these variables might represent countable entities (such as number of workers, items). However, feasible and optimal answers might be fractional (e.g., 2.5 products, 12.8 workers). This is to be expected as we do not impose integer requirements. Forcing some variables to be integers would transform an LP (linear program) to an MIP (mixed integer program), which is out of the scope of this course. Thus, instead of focusing on the real life implications, just try and get used to the fact that all variables in this course are real numbers, not necessarily integers. The solution may coincidentally be integers, which is great, but if not, do not worry. We will be happily producing 33.4 items in a month and carry an inventory of 2.91 items, where 256.34 workers are present in a facility.

2.1 Graphical Method

When there are a tractable number of decision variables, the problem can be solved using the graphical method. Consider the following example.

Example 2.1

The WYNDOR GLASS CO. produces high-quality glass products, including windows and glass doors. It has three plants. Aluminum frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products. Because of declining earnings, top management has decided to revamp the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:

Product 1: An 8-foot glass door with aluminum framing.

Product 2: A 4 x 6 foot double-hung wood-framed window.

Product 1 requires some of the production capacity in Plants 1 and 3, but none in Plant 2. Product 2 needs only Plants 2 and 3. The marketing division has concluded that the company could sell as much of either product as could be produced by these plants. However, because both products would be competing for the same production capacity in Plant 3, it is not clear which mix of the two products would be most profitable. Therefore, an OR team has been formed to study this question. The OR team began by having discussions with upper management to identify management's objectives for the study. These discussions led to developing the following definition of the problem:

Determine what the production rates should be for the two products in order to maximize their total profit, subject to the restrictions imposed by the limited production capacities available in the three plants. (Each product will be produced in batches of 20, so the production rate is defined as the number of batches produced per week.) Any combination of production rates that satisfies these restrictions is permitted, including producing none of one product and as much as possible of the other.

The OR team also identified the data that needed to be gathered:

- Number of hours of production time available per week in each plant for these new products. (Most of the time in these plants already is committed to current products, so the available capacity for the new products is quite limited.)
- Number of hours of production time used in each plant for each batch produced of each new product.
- Profit per batch produced of each new product. (Profit per batch produced was chosen as an appropriate measure after the team concluded that the incremental profit from each additional batch produced would be roughly constant regardless of the total number of batches produced. Because no substantial costs will be incurred to initiate the production and marketing of these new products, the total profit from each one is approximately this profit per batch produced times the number of batches produced.)

The OR team immediately recognized that this was a linear programming problem of the classic product mix type, and the team next undertook the formulation of the corresponding mathematical model.

Required time for two types product to be produced and available time for plants are given in Table 2. Net profits of product 1 and product 2 are \$3,000 and \$5,000, respectively. How would you formulate this problem so as to maximize the profit.

Solution

Decision Variables:

x_1: Batches of Product 1 produced.

Plant	Product 1	Product 2	Production Time Available per week
1	1	0	4
2	0	2	12
3	3	2	18

x_2: Batches of Product 2 produced.

Objective Function:

$$\max 3,000x_1 + 5,000x_2$$

Constraints:

$$\begin{aligned} x_1 \quad\quad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1, \quad x_2 &\geq 0 \end{aligned}$$

⋆ Would we obtain the same solution if we maximized $3x_1 + 5x_2$ instead?

YES! That's what we call *scaling* the objective function.

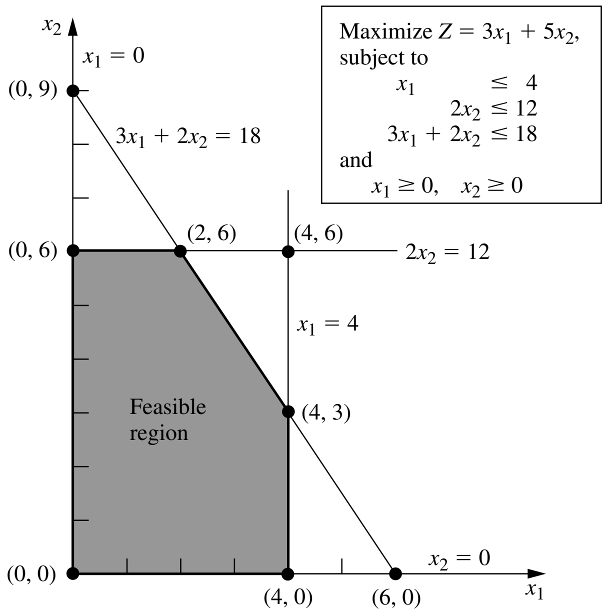
⋆ Would we obtain the same solution if our constraints were the following?

$$\begin{aligned} x_1 \quad\quad &\leq 80 \\ 2x_2 \quad\quad &\leq 240 \\ 3x_1 + 2x_2 &\leq 360 \\ x_1, x_2 &\geq 0 \end{aligned}$$

NO! It's not the same solution but a very similar solution. That's a different way of scaling. We scaled the variables - worked with units instead of batches.

Anyways, we will use the following formulation:

$$\begin{array}{rl} \max & 3x_1 + 5x_2 \\ \text{s.t.} & x_1 \quad\quad \leq 4 \\ & \quad\quad 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, \quad x_2 \geq 0 \end{array}$$



Make sure that you understand how we optimize the objective function using the feasible region in 2D drawing (why 2D?). What is the *range for optimality* for the net profits of first and second product? In other words, provide a range for each coefficient so that the current optimal solution stays optimal.

Pop Question: Can you write 5 inequality constraints with 2 variables so that the feasible region is a

- point?
- line segment?
- line?
- triangle?
- rectangle?
- pentagon?
- hexagon?

If you can, give examples for each.

2.2 Algebraic Simplex Method

FROM NOW ON, ALL VARIABLES ARE SUPPOSED TO BE NONNEGATIVE. YOU CANNOT TREAT A NONPOSITIVE OR UNRESTRICTED VARIABLE THE SAME

WAY IN SIMPLEX METHOD. How to handle variables that are not nonnegative are discussed later in Section 2.4.4.

Example 2.2

$$\begin{array}{rl} \max & 3x_1 + 5x_2 \\ \text{s.t.} & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, \quad x_2 \geq 0 \end{array}$$

Don't worry about the fact that second constraint can be simplified as $x_2 \leq 6$.

2.2.1 *Augmented Form of The Model*

We need the model to be in augmented form (a.k.a. standard form) for the algebraic method to work. The augmented form of the above formulation is as follows:

$$\begin{array}{rl} \max & 3x_1 + 5x_2 \\ \text{s.t.} & x_1 + x_3 = 4 \\ & 2x_2 + x_4 = 12 \\ & 3x_1 + 2x_2 + x_5 = 18 \\ & x_1, \quad x_2, \ x_3, x_4, x_5 \geq 0 \end{array}$$

⋆ Pay attention to the fact that we currently have a feasible solution. It is called initial basic feasible solution (BFS).

⋆ Current solution is $x_1 = 0, x_2 = 0, x_3 = 4, x_4 = 12, x_5 = 18$. From now on, always think in terms of augmented form.

⋆⋆⋆ We will have as many as original number of variables + number of functional constraints. Because each functional constraint will introduce one *slack* variable (if it is a $\leq$ constraint). We will see other forms later.

⋆ Is our current solution optimal? Why/why not?

2.2.2 *Flow of the Algebraic Simplex Method*

$$\begin{array}{rl} \max & z - 3x_1 - 5x_2 + 0x_3 + 0x_4 + 0x_5 = 0 \\ \text{s.t.} & x_1 + x_3 = 4 \\ & 2x_2 + x_4 = 12 \\ & 3x_1 + 2x_2 + x_5 = 18 \end{array}$$

$$x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5 \geq 0$$

⋆ Note that we can omit nonnegativity constraints as all variables are always supposed to be nonnegative for this procedure anyways.

Solution

Current solution is

$$x_1 = 0, x_2 = 0, x_3 = 4, x_4 = 12, x_5 = 18$$

which is not optimal.

See the dynamics at this point. Which variable is willing to increase until which variable hits zero?

Increase x_2. That is x_2 **enters** to the list of nonzero variables, x_4 **leaves** the list of nonzero variables (a.k.a. **BASIS**).
So x_3, x_4, x_5 was in the **BASIS**. But the new **BASIS** is x_2, x_3, x_5.

$$\begin{array}{llcccccl}
\max & z - 3x_1 & + 0\,x_2 & + 0x_3 & + \frac{5}{2}x_4 & + 0x_5 & = 30 \\
\text{s.t.} & x_1 & & + x_3 & & & = 4 \\
& & x_2 & & + \frac{1}{2}x_4 & & = 6 \\
& 3x_1 & & & - x_4 & + x_5 & = 6
\end{array}$$

Current solution is

$$x_1 = 0, x_2 = 6, x_3 = 4, x_4 = 0, x_5 = 6$$

which is not optimal. Why?

⋆ NOTE: Variables in the BASIS are called *basic variables*, and those not in the basis are called *nonbasic variables.*

⋆ How can you tell if a variable is nonbasic?

⋆ How can you tell the values of basic variables?

Increase x_1. x_1 **enters** the **BASIS**, x_5 **leaves** the **BASIS**.

$$\begin{array}{llcccccl}
\max & z + 0x_1 & + 0x_2 & + 0x_3 & + \frac{3}{2}x_4 & + x_5 & = 36 \\
\text{s.t.} & & + & x_3 & + \frac{1}{3}x_4 & - \frac{1}{3}x_5 & = 2 \\
& & x_2 & + & \frac{1}{2}x_4 & & = 6 \\
& x_1 & & - & \frac{1}{3}x_4 & + \frac{1}{3}x_5 & = 2
\end{array}$$

Current solution is

$$x_1 = 2, x_2 = 6, x_3 = 2, x_4 = 0, x_5 = 0, z = 36$$

which is optimal. Why?

This optimality is denoted as:

$x_1^* = 2, x_2^* = 6, x_3^* = 2, x_4^* = 0, x_5^* = 0, z^* = 36$

OPTIMAL BASIS is x_1, x_2, x_3.

⋆⋆⋆ FLOW OF SIMPLEX METHOD ⋆⋆⋆

(1) Find the entering variable (the one with most negative reduced cost, i.e., row zero coefficient)

(2) Find the pivot number that provides *minimum ratio* of right hand side divided by constraint coefficient

(3) Identify the leaving variable

(4) Make the pivot number one, by row multiplication

(5) Try to make constraint coefficients zero in other rows for the entering variable in an effort to produce a new identity matrix

(6) You will notice the leaving variable's identity matrix column disappears and right hand side is always nonnegative (if not, minimum ratio test was not done correctly)

(7) Make the reduced cost of entering variable zero through basic row operations (other basic variables' reduced costs has to remain zero for the obvious reason)

(8) You will notice the objective has improved. Double check if you have a new identity matrix under the new basis, and if reduced cost of these basic variables are all zero.

Example 2.3 Solve the following problem in class.

$$\begin{aligned} \min\ & -2x_1 + 6x_2 - 7x_3 \\ \text{s.t.}\ & x_1 + x_2 + 2x_3 \le 1 \\ & x_1 + 4x_3 \le 2 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Solution

$$\begin{array}{llllllll} \max & -z & -2x_1 & +6x_2 & -7x_3 & +0x_4 & +0x_5 & = 0 \\ \text{s.t.} & & x_1 & +x_2 & +2x_3 & +x_4 & & = 1 \\ & & x_1 & & +4x_3 & & +x_5 & = 2 \end{array}$$

x_1, x_2, x_3 are nonbasic variables.
$x_4 = 1, x_5 = 2, z = 0$

x_3 enters, x_5 leaves.

$$\begin{array}{llllllll} \max & -z & -\frac{1}{4}x_1 & +6x_2 & & & +\frac{7}{4}x_5 & = \frac{7}{2} \\ \text{s.t.} & & \frac{1}{2}x_1 & +x_2 & & +x_4 & -\frac{1}{2}x_5 & = 0 \\ & & \frac{1}{4}x_1 & & +x_3 & & +\frac{1}{4}x_5 & = \frac{1}{2} \end{array}$$

x_1, x_2, x_5 are nonbasic variables.
$x_4 = 0$, $x_3 = \frac{1}{2}$, $z = -\frac{7}{2}$

x_1 enters, x_4 leaves.

$$\begin{array}{llllllll} \max & -z & & +\frac{13}{2}x_2 & & +\frac{1}{2}x_4 & +\frac{3}{2}x_5 & = \frac{7}{2} \\ \text{s.t.} & & x_1 & +2x_2 & & +2x_4 & -x_5 & = 0 \\ & & & -\frac{1}{2}x_2 & +x_3 & -\frac{1}{2}x_4 & +\frac{1}{2}x_5 & = \frac{1}{2} \end{array}$$

x_4, x_2, x_5 are nonbasic variables.
$x_1 = 0$, $x_3 = \frac{1}{2}$, $z = -\frac{7}{2}$
Optimal solution is found! $x_4^*, x_2^*, x_5^* = 0, x_1^* = 0, x_3^* = \frac{1}{2}, z^* = -\frac{7}{2}$

$\star\star\star$ Nonbasic variables are always zero. However, basic variables are not necessarily nonzero! Above example illustrates that case, also known as **degeneracy**.

NOTES

1. Each variable is designated as basic or nonbasic (there are a total of n variables).
2. Number of basic variables **always equals** the number of constraints (there are m BV's).
3. Nonbasic variables are **always** set to zero (there are $n - m$ NBV's).
4. System can be solved for basic variables using m equations and m unknowns. Then we end up with a **Basic Solution** (a.k.a. corner point (CP) solution).
5. If basic variables are nonnegative then we have a **Basic Feasible Solution** (BFS).
6. In Simplex Method, we are exploring BFS's (a.k.a. corner point feasible (CPF) solutions).

QUESTIONS

(1) In Example 2.2 in your notes, enumerate all basic solutions. Classify these solutions as feasible or infeasible.

(2) In Example 2.3 in your notes, enumerate all basic solutions. Classify these solutions as feasible or infeasible.

(3) Consider the following problem.

$$
\begin{aligned}
\max \quad & z = 3x_1 + 2x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \leq 6 \\
& x_1 + 2x_2 \leq 6 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

(a) Use the graphical method to solve this problem. Circle all the corner points on the graph.

(b) For each CPF solution, identify the pair of constraint boundary equations it satisfies.

(c) For each CPF solution, identify its adjacent CPF solutions.

(d) Calculate z for each CPF solution. Use this information to identify an optimal solution.

(e) Describe graphically what the simplex method does step by step to solve the problem.

$x_1^* = 2, x_2^* = 2, z^* = 10$

(4) Work through the simplex method (in algebraic form) step by step to solve the following problem.

$$
\begin{aligned}
\min \quad & z = -x_1 - 2x_2 - 4x_3 \\
\text{s.t.} \quad & 3x_1 + x_2 + 5x_3 \leq 10 \\
& x_1 + 4x_2 + x_3 \leq 8 \\
& 2x_1 + 2x_3 \leq 7 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$x_1^* = 0, x_2^* = 30/19, x_3^* = 32/19, z^* = -188/19$

2.3 Simplex Method in Tabular Form / Tableau Format

Example 2.4 Wyndor Glass Co. Problem

$$\begin{array}{lllllll}
\max & z - 3x_1 & -5x_2 & +0x_3 & +0x_4 & +0x_5 & = 0 \\
\text{s.t.} & x_1 & & +x_3 & & & = 4 \\
 & & 2x_2 & & +x_4 & & = 12 \\
 & 3x_1 & +2x_2 & & & +x_5 & = 18 \\
 & x_1 & , x_2 & \geq 0 & & &
\end{array}$$

Solution

x_3, x_4, x_5 are basic variables.

Iteration 0

z	x_1	x_2	x_3	x_4	x_5	RHS
1	-3	-5	0	0	0	0
0	1	0	1	0	0	4
0	0	2	0	1	0	12
0	3	2	0	0	1	18

Iteration 1

z	x_1	x_2	x_3	x_4	x_5	RHS
1	-3	0	0	$\frac{5}{2}$	0	30
0	1	0	1	0	0	4
0	0	1	0	$\frac{1}{2}$	0	6
0	3	0	0	-1	1	6

Iteration 2

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	$\frac{3}{2}$	1	36
0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
0	0	1	0	$\frac{1}{2}$	0	6
0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Note the changes (especially in the z column in case of a minimization problem. Before we proceed, let's make sure that you can solve relatively easy questions.

Example 2.5

$$\begin{aligned} \min\ & 2x_2 - 6x_1 \\ \text{s.t.}\ & 2x_1 - x_2 \le 2 \\ & x_1 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

2.4 Ill-Posed Cases

Note that all ties are broken arbitrarily in simplex method. However, there are cases that might riddle you with exceptions or abrupt zeros. Below we summarize all possible ill-posed scenarios that you might come across.

2.4.1 *Alternative Optima*

If we have alternative solutions that give the same optimal objective function value, then there is alternative optima.

Example 2.6

$$
\begin{aligned}
\max\ & 3x_1 + 5x_2 + 4x_3 \\
\text{s.t.}\ & x_1 \le 5 \\
& 2x_2 + 2x_3 \le 12 \\
& 3x_1 + 2x_2 + x_3 \le 18 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

Iteration 0:

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	-3	-5	-4	0	0	0	0
0	1	0	0	1	0	0	5
0	0	2	2	0	1	0	12
0	3	2	1	0	0	1	18

Iteration 1:

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	-3	0	1	0	$\frac{5}{2}$	0	30
0	1	0	0	1	0	0	5
0	0	1	1	0	$\frac{1}{2}$	0	6
0	3	0	-1	0	-1	1	6

Iteration 2:

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	$\boxed{0}$	0	$\frac{3}{2}$	1	36
0	0	0	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{-1}{3}$	3
0	0	1	1	0	$\frac{1}{2}$	0	6
0	1	0	$\frac{-1}{3}$	0	$\frac{-1}{3}$	$\frac{1}{3}$	2

This final tableau is optimal. However, x_3 can enter the basis improving the objective with a rate of zero, that is staying at the same objective function value.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	$\boxed{0}$	0	0	$\frac{3}{2}$	1	36
0	0	$\frac{-1}{3}$	0	1	$\frac{1}{6}$	$\frac{-1}{3}$	1
0	0	1	1	0	$\frac{1}{2}$	0	6
0	1	$\frac{1}{3}$	0	0	$\frac{-1}{6}$	$\frac{1}{3}$	4

⋆⋆⋆ When the reduced costs of a nonbasic variable is zero in the **final tableau**, we can say that there is alternative optima.

⋆ Reduced cost is **row zero values** in the tableau. That is the row that corresponds to the *objective function* after row operations in each iteration.

2.4.2 *Degeneracy*

Example 2.7 The following tableau presents a case of degeneracy:

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	-3	0	0	$\frac{3}{2}$	1	23
0	0	$\frac{1}{3}$	0	1	$\frac{1}{6}$	$\frac{-1}{3}$	$\boxed{0}$
0	0	1	1	0	$\frac{1}{2}$	0	3
0	1	$\frac{1}{3}$	0	0	$\frac{-1}{6}$	$\frac{1}{3}$	2

⋆ If a basic variable equals zero, then there is a degenerate solution.

⋆ Question: Would this still be degenerate if the coefficient of x_2 in row zero was positive? Or if constraint 1 coefficient of x_2 was negative?

Answer: Looking at the definition, yes! Think about how you would provide that alternative representation of the same solution...

2.4.3 *Unboundedness*

Example 2.8

$$\begin{aligned} \max\ & 3x_1 + 5x_2 + 3x_3 \\ \text{s.t.}\ & x_1 - 2x_3 \leq 4 \\ & 2x_2 - x_3 \leq 12 \\ & 3x_1 + 2x_2 - x_3 \leq 18 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Final tableau:

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	$\boxed{\frac{-11}{2}}$	0	$\frac{3}{2}$	1	36
0	0	0	$\boxed{-2}$	1	$\frac{1}{3}$	$\frac{-1}{3}$	2
0	0	1	$\boxed{\frac{-1}{2}}$	0	$\frac{1}{2}$	0	6
0	1	0	$\boxed{0}$	0	$\frac{-1}{3}$	$\frac{1}{3}$	2

x_3 enters but there is no leaving variable. Unbounded problem!

⋆ The problem is unbounded, if for a variable with negative reduced cost (not necessarily the one with most negative reduced cost), ratio test does not provide a value - that is, constraint coefficients are nonpositive for that variable.

⋆ Question: Can you provide an answer whose objective function value is 600? Or 6000?

2.4.4 *Dealing with Unrestricted or Nonpositive Variables*

How do we handle variables that are not nonnegative?

Suppose we have the following problem.

$$\begin{aligned} \max\ & 3x_1 + 2x_2 \\ \text{s.t.}\ & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 16 \\ & x_1 \geq 0 \\ & x_2 \leq -10 \end{aligned}$$

We already know how to use simplex, and **simplex can only solve LP's with nonnegative variables**.

What substitution would you use to make all variables nonnegative?

- $x_2^+ = -x_2$. In this case, we would have $x_2^+ \geq 0$ and another functional constraint $x_2^+ \geq 10$.
- $x_2^+ = -x_2 - 10$. In this case, we would only have $x_2^+ \geq 0$, which is better than the alternative above.

Substitute $x_2^+ = -x_2 - 10$, that is $\boxed{x_2 = -x_2^+ - 10}$ $x_2^+ \geq 0$

Note: Original x_2 disappears in the updated formulation. If you need to find it, you need to use x_2^+ to compute it using the relationship $x_2 = -x_2^+ - 10$.

$$\begin{aligned} \max\ & 3x_1 - 2x_2^+ - 20 \\ \text{s.t.}\ & x_1 \leq 4 \\ & -2x_2^+ - 20 \leq 12 \\ & 3x_1 - 2x_2^+ - 20 \leq 16 \\ & x_1 \geq 0 \\ & x_2^+ \geq 0 \end{aligned}$$

Suppose we have the following problem.

$$\begin{aligned} \max\ & 3x_1 + 2x_2 \\ \text{s.t.}\ & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 16 \\ & x_1 \geq 0 \\ & x_2 \quad \text{urs.} \end{aligned}$$

What substitution would you use to make all variables nonnegative?

<u>Substitute</u> $\boxed{x_2 = x_2^+ - x_2^-}$ $x_2^+, x_2^- \geq 0$

Note: Original x_2 disappears in the updated formulation. If you need to find it, you need to use x_2^+ and x_2^- to compute it using the relationship $x_2 = x_2^+ - x_2^-$.

$$\begin{aligned} \max\ & 3x_1 + 2x_2^+ - 2x_2^- \\ \text{s.t.}\ & x_1 \leq 4 \\ & 2x_2^+ - 2x_2^- \leq 12 \\ & 3x_1 + 2x_2^+ - 2x_2^- \leq 16 \\ & x_1, x_2^+, x_2^- \geq 0 \end{aligned}$$

Before we proceed, let's make sure that you can use all techniques you have learned thus far.

Example 2.9

$$\begin{aligned} \min\ & 5x_1 + 3x_2 \\ \text{s.t.}\ & x_1 \geq -8 \\ & 2x_1 + 4x_2 \leq 12 \\ & x_1 \text{ urs}, x_2 \leq -6 \end{aligned}$$

2.5 Finding an Initial Basic Feasible Solution

We have seen how to deal with minimization or maximization problems that have $\leq$ constraints. Note that all these problems initially had nonnegative right hand sides as well. What we have learned so far will not be useful to solve the problems that utilize either one of the following constraints:

- $4x_1 + 2x_2 + x_3 \geq 24$
- $x_1 - 2x_2 - 5x_3 \leq -24$
- $3x_1 + x_2 + 3x_3 = 24$ (What's wrong with assuming x_2 is a BV?)
- $2x_1 - 3x_2 + x_3 = -24$

First, make sure that you understand why we cannot handle these. In other words, why is it challenging to find an initial BFS in these cases?

Fortunately, the way we handle such situations is pretty standard. There are two alternative ways that help us solve any of these 4 cases: Big-M method and Two phase method.

2.5.1 *Big-M Method*

Example 2.10

$$\begin{aligned} \max\ & -600x_1 - 500x_2 - 700x_3 \\ \text{s.t.}\ & 4x_1 + 2x_2 + 5x_3 \geq 24 \\ & 5x_1 + 6x_2 + 2x_3 \geq 35 \\ & 3x_1 + 3x_2 + 4x_3 \geq 30 \\ & x_i \geq 0, \quad i = 1, 2, 3 \end{aligned}$$

Solution

$$\begin{aligned} \max\ & -600x_1 - 500x_2 - 700x_3 \\ \text{s.t.}\ & 4x_1 + 2x_2 + 5x_3 - x_4 = 24 \\ & 5x_1 + 6x_2 + 2x_3 - x_5 = 35 \\ & 3x_1 + 3x_2 + 4x_3 - x_6 = 30 \\ & x_i \geq 0, \quad i = 1, 2, \ldots, 6 \end{aligned}$$

$$\begin{aligned} \max\ & -600x_1 - 500x_2 - 700x_3 - Mx'_4 - Mx'_5 - Mx'_6 \\ \text{s.t.}\ & 4x_1 + 2x_2 + 5x_3 - x_4 + x'_4 = 24 \\ & 5x_1 + 6x_2 + 2x_3 - x_5 + x'_5 = 35 \\ & 3x_1 + 3x_2 + 4x_3 - x_6 + x'_6 = 30 \\ & x_i \geq 0, \quad i = 1, 2, \ldots, 6 \\ & x'_i \geq 0, \quad i = 4, 5, 6 \end{aligned}$$

What we did is we penalized these *artificial variables* (x'_4, x'_5, x'_6) heavily in the objective. If possible it will find a solution where these artificial variables are zeros. Otherwise, if these artificial variables are not zero, the objective function value will be a function of M.

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	600	500	700	0	0	0	M	M	M	0
0	4	2	5	-1	0	0	1	0	0	24
0	5	6	2	0	-1	0	0	1	0	35
0	3	3	4	0	0	-1	0	0	1	30

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	$600-12M$	$500-11M$	$700-11M$	M	M	M	0	0	0	$-89M$
0	4	2	5	-1	0	0	1	0	0	24
0	5	6	2	0	-1	0	0	1	0	35
0	3	3	4	0	0	-1	0	0	1	30

Next, you have to use simplex method until all reduced costs are nonnegative.

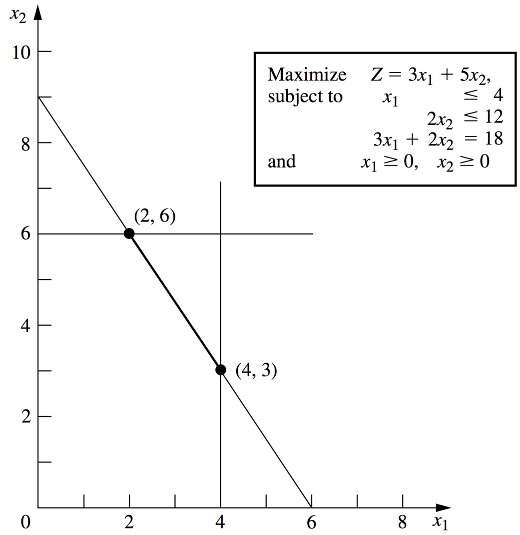
⋆ Which variable enters first?

ANSWER: x_1 because M is a very large number!

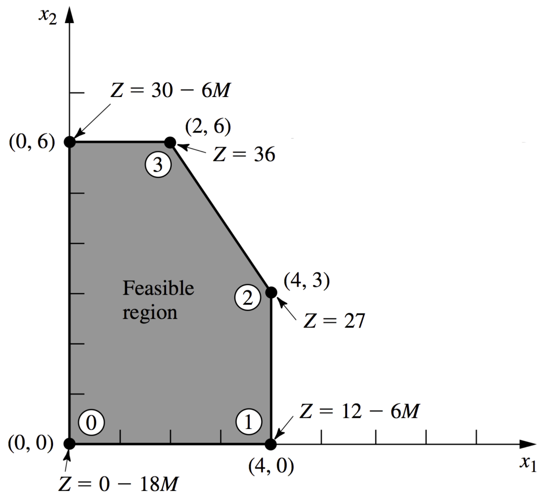
⋆⋆⋆ If the optimal objective function value is a function of M, that shows your problem does not have a feasible solution! The problem is then **infeasible**!

Graphical Illustration of the Big-M Method

Consider the Wyndor Glass Co. problem defined in Example 2.1 (formulation in Example 2.2). Suppose the third functional constraint becomes an equality constraint. Then the feasible region becomes the line segment between (2, 6) and (4, 3) as shown in the graph below.



If we solve this problem using Big-M method, the objective function for each basic feasible solution (which are indeed not feasible) are provided in the picture below. Note that such solutions will always have an undesirable objective function value (less than or equal to $-M$). Below is the solution of this problem using Big-M method.



Example 2.11

$$\begin{aligned} &\max\ 3x_1 + 5x_2 \\ &\text{s.t.}\quad x_1 \leq 4 \\ &\qquad 2x_2 \leq 12 \\ &\qquad 3x_1 + 2x_2 = 18 \\ &\qquad x_i \geq 0, \quad i = 1, 2 \end{aligned}$$

Solution

$$\begin{aligned} &\max\ z - 3x_1 - 5x_2 + Mx_5 \\ &\text{s.t.}\quad x_1 + x_3 = 4 \\ &\qquad 2x_2 + x_4 = 12 \\ &\qquad 3x_1 + 2x_2 + x_5' = 18 \\ &\qquad x_i \geq 0, \quad i = 1, 2, \ldots, 5 \end{aligned}$$

Note that x_5' is an artificial variable here. x_3 and x_4 on the other hand are slack variables that already help us with the initial BFS.

z	x_1	x_2	x_3	x_4	x_5'	RHS
1	-3	-5	0	0	M	0
0	1	0	1	0	0	4
0	0	2	0	1	0	12
0	3	2	0	0	1	18

z	x_1	x_2	x_3	x_4	x_5'	RHS
1	$-3-3M$	$-5-2M$	0	0	0	$-18M$
0	1	0	1	0	0	4
0	0	2	0	1	0	12
0	3	2	0	0	1	18

x_1 enters, x_3 leaves.

z	x_1	x_2	x_3	x_4	x_5'	RHS
1	0	$-5-2M$	$3M+3$	0	0	$-6M+12$
0	1	0	1	0	0	4
0	0	2	0	1	0	12
0	0	2	-3	0	1	6

x_2 enters, x_5' leaves.

z	x_1	x_2	x_3	x_4	x_5'	RHS
1	0	0	$\frac{-9}{2}$	0	$\frac{5+2M}{2}$	27
0	1	0	1	0	0	4
0	0	0	3	1	-1	6
0	0	1	$\frac{-3}{2}$	0	$\frac{1}{2}$	3

x_3 enters, x_4 leaves.

z	x_1	x_2	x_3	x_4	x_5'	RHS
1	0	0	0	$\frac{3}{2}$	$M+1$	36
0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
0	0	0	1	$\frac{1}{3}$	$\frac{-1}{3}$	2
0	0	1	0	$\frac{1}{2}$	0	6

The optimal solution is $x_1^* = 2, x_2^* = 6, x_3^* = 2, x_4^* = 0, x_5'^* = 0, z^* = 36$.

More Examples

Example 2.12 Solve the following problem in class.

$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 4 \\ & x_1 + x_2 = 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution

$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 - Mx_4' \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 = 4 \\ & x_1 + x_2 + x_4' = 3 \\ & x_1, x_2, x_3, x_4' \geq 0 \end{aligned}$$

Initial basic feasible solution: (0,0,4,3)

z	x_1	x_2	x_3	x_4'	RHS
1	-2	-3	0	M	0
0	1	2	1	0	4
0	1	1	0	1	3

z	x_1	x_2	x_3	x'_4	RHS
1	$-0.5M-0.5$	0	$0.5M+1.5$	0	$6-M$
0	0.5	1	0.5	0	2
0	0.5	0	-0.5	1	1

z	x_1	x_2	x_3	x'_4	RHS
1	0	0	1	$1+M$	7
0	0	1	1	-1	1
0	1	0	-1	2	2

Optimal Solution (x_1^*, x_2^*)=(2,1) and $z^* = 7$.

2.5.2 *Two Phase Method*

Example 2.13

$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \leq 40 \\ & 2x_1 + x_2 - x_3 \geq 10 \\ & -x_2 + x_3 \geq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution

$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 = 40 \\ & 2x_1 + x_2 - x_3 - x_5 = 10 \\ & -x_2 + x_3 - x_6 = 10 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

<u>**PHASE 1**</u>

$$\begin{aligned} & \min x_5' + x_6' \\ & \text{s.t. } x_1 + x_2 + x_3 + x_4 = 40 \\ & \qquad 2x_1 + x_2 - x_3 - x_5 + x_5' = 10 \\ & \qquad -x_2 + x_3 - x_6 + x_6' = 10 \\ & \qquad x_1, x_2, x_3, x_4, x_5, x_5', x_6, x_6' \geq 0 \end{aligned}$$

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_5	x'_6	RHS
-1	0	0	0	0	0	0	1	1	0
0	1	1	1	1	0	0	0	0	40
0	2	1	-1	0	-1	0	1	0	10
0	0	-1	1	0	0	-1	0	1	10

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_5	x'_6	RHS
-1	-2	0	0	0	1	1	0	0	-20
0	1	1	1	1	0	0	0	0	40
0	2	1	-1	0	-1	0	1	0	10
0	0	-1	1	0	0	-1	0	1	10

Entering and leaving variables would be x_1 and x'_5 respectively:

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_5	x'_6	RHS
-1	0	1	-1	0	0	1	1	0	-10
0	0	0.5	1.5	1	0.5	0	-0.5	0	35
0	1	0.5	-0.5	0	-0.5	0	0.5	0	5
0	0	-1	1	0	0	-1	0	1	10

Entering and leaving variables would be x_3 and x'_6 respectively:

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_5	x'_6	RHS
-1	0	0	0	0	0	0	1	1	0
0	0	2	0	1	0.5	1.5	-0.5	-1.5	20
0	1	0	0	0	-0.5	-0.5	0.5	0.5	10
0	0	-1	1	0	0	-1	0	1	10

The optimal value of the Phase I problem is 0. Therefore, the original problem is feasible, and a basic feasible solution is $x_1 = 10, x_3 = 10, x_4 = 20, x_2 = x_5 = x_6 = 0$. Next, we start Phase II by simply omitting artificial variables x'_5 and x'_6.

PHASE 2

The initial tableau is the last Phase I tableau where artificial variables taken away. Also note that we use the original problem's objective function row.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	-2	-3	-1	0	0	0	0
0	0	2	0	1	0.5	1.5	20
0	1	0	0	0	-0.5	-0.5	10
0	0	-1	1	0	0	-1	10

Why we did this entire phase I is to obtain this BFS above. Notice that there is identity matrix structure in the constraints with nonnegative right hand sides. These variables' reduced costs were supposed to be zero but even though that is not the case above, that is very easy to obtain using row operations with no drawbacks.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	-4	0	0	-1	-2	30
0	0	2	0	1	0.5	1.5	20
0	1	0	0	0	-0.5	-0.5	10
0	0	-1	1	0	0	-1	10

Entering and leaving variables would be x_2 and x_4 respectively.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	0	2	0	1	70
0	0	1	0	0.5	0.25	0.75	10
0	1	0	0	0	-0.5	-0.5	10
0	0	0	1	0.5	0.25	-0.25	20

Thus, the optimal value is $z^* = 70$, and the optimal solution is $x_1^* = x_2^* = 10, x_3^* = 20, x_4^* = x_5^* = x_6^* = 0$.

Example 2.14 Solve the following problem in class.

$$\begin{aligned} \min \quad & z = x_1 + x_2 + 3x_3 + x_4 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + 6x_4 \geq 14 \\ & 3x_1 + x_2 + 2x_3 - 7x_4 = -11 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Solution

PHASE 1

$$\begin{aligned} & \min x_5' + x_6' \\ & \text{s.t.} \ 2x_1 + 3x_2 + 6x_4 - x_5 + x_5' = 14 \\ & \quad -3x_1 - x_2 - 2x_3 + 7x_4 + x_6' = 11 \\ & \quad x_1, x_2, x_3, x_4, x_5, x_5', x_6' \geq 0 \end{aligned}$$

Why don't we introduce the artificial variable to the original constraint as is, so that it reads $3x_1 + x_2 + 2x_3 - 7x_4 + x_6' = -11$?

z	x_1	x_2	x_3	x_4	x_5	x_5'	x_6'	RHS
-1	0	0	0	0	0	1	1	0
0	2	3	0	6	-1	1	0	14
0	-3	-1	-2	7	0	0	1	11

z	x_1	x_2	x_3	x_4	x_5	x_5'	x_6'	RHS
-1	1	-2	2	-13	1	0	0	-25
0	2	3	0	6	-1	1	0	14
0	-3	-1	-2	7	0	0	1	11

z	x_1	x_2	x_3	x_4	x_5	x'_5	x'_6	RHS
-1	$\frac{-32}{7}$	$\frac{-27}{7}$	$\frac{-12}{7}$	0	1	0	$\frac{13}{7}$	$\frac{-32}{7}$
0	$\frac{32}{7}$	$\frac{27}{7}$	$\frac{12}{7}$	0	-1	1	$\frac{-6}{7}$	$\frac{32}{7}$
0	$\frac{-3}{7}$	$\frac{-1}{7}$	$\frac{-2}{7}$	1	0	0	$\frac{1}{7}$	$\frac{11}{7}$

z	x_1	x_2	x_3	x_4	x_5	x'_5	x'_6	RHS
-1	0	0	0	0	0	1	1	0
0	1	$\frac{27}{32}$	$\frac{3}{8}$	0	$\frac{-7}{32}$	$\frac{7}{32}$	$\frac{-3}{16}$	1
0	0	$\frac{7}{32}$	$\frac{-1}{8}$	1	$\frac{-3}{32}$	$\frac{3}{32}$	$\frac{1}{16}$	2

PHASE 2

z	x_1	x_2	x_3	x_4	x_5	RHS
-1	1	1	3	1	0	0
0	1	$\frac{27}{32}$	$\frac{3}{8}$	0	$\frac{-7}{32}$	1
0	0	$\frac{7}{32}$	$\frac{-1}{8}$	1	$\frac{-3}{32}$	2

z	x_1	x_2	x_3	x_4	x_5	RHS
-1	0	$\frac{-1}{16}$	$\frac{11}{4}$	0	$\frac{5}{16}$	-3
0	1	$\frac{27}{32}$	$\frac{3}{8}$	0	$\frac{-7}{32}$	1
0	0	$\frac{7}{32}$	$\frac{-1}{8}$	1	$\frac{-3}{32}$	2

z	x_1	x_2	x_3	x_4	x_5	RHS
-1	$\frac{2}{27}$	0	$\frac{25}{9}$	0	$\frac{8}{27}$	$\frac{-79}{27}$
0	$\frac{32}{27}$	1	$\frac{4}{9}$	0	$\frac{-7}{27}$	$\frac{32}{27}$
0		0		1		$\frac{47}{27}$

The solution is optimal. This tableau corresponds to the BFS $x^* = (0, \frac{32}{27}, 0, \frac{47}{27}, 0)$ with $z^* = \frac{79}{27}$. Note that we found $-79/27$ as our objective function value at optimality but that was for the maximization problem (all coefficients' signs were flipped in the objective). We flip the sign of the objective function value for the original minimization problem's objective function value.

NOTES

- If the original problem has no feasible solutions then either Big-M method or phase I of the two phase method yields a final solution that has at least one artificial variable greater than zero. Otherwise, they all equal zero and the problem is feasible.
- Make sure that you understand how to handle variables that are nonpositive or unrestricted (urs).
- It is virtually impossible to tell if Big-M method or two phase method would yield an easier path to the optimal solution. You have to know how both methods work.

2.6 Exercises

(1) Solve the following problem.

$$\begin{aligned} \min \quad & z = 2x_1 + 3x_2 + x_3 \\ \text{s.t.} \quad & x_1 + 4x_2 + 2x_3 \geq 8 \\ & -3x_1 - 2x_2 \leq -6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

z^*	x_1^*	x_2^*	x_3^*

(2) Consider Example 2.10 in your notes:

- Solve the problem using Big-M method and provide the optimal solution.
- Solve the problem using two phase method and provide the optimal solution.

(3) Find the optimal solution to the problem below.

$$\begin{aligned} \max \quad & 2x_1 + 5x_2 + 3x_3 + x_4 \\ \text{subject to} \quad & x_1 - 2x_2 + x_3 \geq 20 \\ & 2x_1 + 4x_2 + x_3 = 50 \\ & x_4 + x_1 \leq 10 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

z^*	x_1^*	x_2^*	x_3^*	x_4^*

(4) Provide explanations for your answers below.

(a) The best corner point feasible solution is an optimal solution. True or false?

(b) Convert the following inequality constraint into an **equality constraint** with **non-negative variables**.

$$\begin{aligned} & x_1 - x_2 + 2x_3 \geq 1 \\ & x_1 \geq 0, x_2 \leq 0, x_3 \text{ unrestricted} \end{aligned}$$

(c) In each iteration of the simplex method, the value of the objective function strictly improves. True or false?

(d) How do you detect if a Linear Programming model is infeasible or not?

(e) If the feasible region of a LP problem (maximization) with 2 variables is unbounded, then the value of the objective function can be increased indefinitely. True or false?

(5) Solve the following Linear Program using two phase method.

$$\begin{aligned} \min \quad & 3x_1 + x_3 - 3x_2 \\ \text{subject to } & x_1 + 2x_2 - x_3 \geq 5 \\ & -3x_1 - x_2 + x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

z^*	x_1^*	x_2^*	x_3^*

(6) Is it possible for an optimization problem to have an unbounded feasible region and alternative optima at the same time? If it is possible, give an example. If not, explain why not.

(7) Provide the optimal solution and the optimal objective function value for the following problem using the Big-M method.

$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 4 \\ & x_1 + x_2 = 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

z^*	x_1^*	x_2^*

2.7 Further Reading and Exercises

The reader is refereed to Chapter 4 in the textbook, *Introduction to Operations Research* by Hillier and Lieberman.

Chapter 3

Theory of Simplex Method and Revised Simplex Method (Weeks 8-9)

The roots of the revised simplex method stems from the fundamentals and row operations of the simplex algorithm. Keep in mind that **for any set of basic variables**, a legitimate simplex tableau can be constructed from the original problem as follows:

z	Basic Variables	Nonbasic Variables	RHS	
1	$-c_B$	$-c_N$	0	ORIGINAL
$\vec{0}$	B	N	b	PROBLEM

$$\Downarrow$$

z	Basic Variables	Nonbasic Variables	RHS	
1	$\vec{0}$	$c_B B^{-1} N - c_N$	$c_B B^{-1} b$	CORRESPONDING
$\vec{0}$	I	$B^{-1} N$	$B^{-1} b$	SIMPLEX
				TABLEAU

You can always use $c_B B^{-1} A_j - c_j$ and $B^{-1} A_j$ in the corresponding simplex tableau for any variable j, regardless of being basic or nonbasic.

Consider the Wyndor Glass Co. Example in the following **augmented form**:

$$
\begin{array}{rlllll}
\max & 3x_1 + & 5x_2 & & & \\
\text{s.t.} & x_1 & + & x_3 & & = 4 \\
& & 2x_2 + & x_4 & & = 12 \\
& 3x_1 + & 2x_2 & + & x_5 & = 18 \\
& x_1, & x_2, & x_3, & x_4, x_5 & \geq 0
\end{array}
$$

First iteration for x_3, x_4, x_5 in the basis:

Basic Variables: $x_B = [x_3, x_4, x_5]$

Nonbasic Variables: $x_N = [x_1, x_2]$

Coefficients of basic variables in objective function: $c_B = [0\ 0\ 0]$

Coefficients of nonbasic variables in objective function: $c_N = [3\ 5]$

Coefficients of basic variables in constraints:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Coefficients of nonbasic variables in constraints:

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1}N = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \quad B^{-1}b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

Optimality check:

$$c_B B^{-1} N - c_N = [0\ 0\ 0] \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} - [3\ 5]$$
$$= [-3\ \ -5]$$

I recommend that you try and picture the tableau that corresponds to that basis x_3, x_4, x_5.

Entering variable is x_2.

Minimum Ratio Test: min { $\frac{4}{0}$=unidentified , $\frac{12}{2}$=6 , $\frac{18}{2}$=9 }= 6

Leaving Variable is x_4.

Second iteration for basis x_3, x_2, x_5:

$x_B = [x_3, x_2, x_5]$

$x_N = [x_1, x_4]$

$c_B = [0\ 5\ 0]$

$c_N = [3\ 0]$

$$B = \begin{bmatrix} 1\ 0\ 0 \\ 0\ 2\ 0 \\ 0\ 2\ 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} 1\ 0 \\ 0\ 1 \\ 3\ 0 \end{bmatrix} \quad B^{-1}N = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

Optimality check:

$$c_B B^{-1} N - c_N = [0\ 5\ 0] \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 3 & -1 \end{bmatrix} - [3\ 0]$$
$$= [-3\ \ \frac{5}{2}]$$

Entering variable is x_1.

Minimum Ratio Test: min { $\frac{4}{1}$=4 , $\frac{6}{0}$=unidentified , $\frac{6}{3}$=2 }=2

Leaving Variable is x_5.

<u>THIRD ITERATION FOR BASIS x_3, x_2, x_1:</u>

x_B= [x_3 , x_2 , x_1]

x_N= [x_5 , x_4]

c_B= [0 5 3]

c_N= [0 0]

$$B = \begin{bmatrix} 1\ 0\ 1 \\ 0\ 2\ 0 \\ 0\ 2\ 3 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & \frac{-1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{-1}{3} & \frac{1}{3} \end{bmatrix}$$

$$N = \begin{bmatrix} 0\ 0 \\ 0\ 1 \\ 1\ 0 \end{bmatrix} \quad B^{-1}N = \begin{bmatrix} \frac{-1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{-1}{3} \end{bmatrix} \quad B^{-}b = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

Optimality check:

$$\begin{aligned} c_B B^{-1} N - c_N &= [0\ 5\ 3] \begin{bmatrix} \frac{-1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{-1}{3} \end{bmatrix} - [0\ 0] \\ &= [1\ \ \frac{3}{2}] \end{aligned}$$

It is the optimal solution!

Objective Function Value: $z^* = c_B B^{-1} b = [0\ 5\ 3] \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 36$

Make sure that you can report the values of variables x_1 through x_5.

3.1 Revised Simplex Method

Initialization: Same as for the original simplex method.

Iteration:

Step 1: Determine the entering basic variable: Same as for the original simplex method.

Step 2: Determine the leaving basic variable: Same as for the original simplex method, except calculate only the numbers required to do this - the coefficients of the entering basic variable in every equation but Eq. (0), and then, for each strictly positive coefficient, the right-hand side of that equation.

Step 3: Determine the new BF solution: Derive B^{-1} and set $x_B = B^{-1}b$.

Optimality test: Same as for the original simplex method, except calculate only the numbers required to do this test, i.e., the coefficients of the nonbasic variables in Eq. (0).

3.2 Updating B^{-1} Without Matrix Inversion

To describe this method formally,

let x_k be the entering basic variable,

a'_{ik}= coefficient of x_k in current Eq. (i), for $i = 1, 2, ..., m$ (calculated in step 2 of an iteration),

r = equation index for the leaving basic variable.

$$(B^{-1}_{new})_{ij} = \begin{cases} (B^{-1}_{old})_{ij} - \frac{a'_{ik}}{a'_{rk}}(B^{-1}_{old})_{rj} & if \quad i \neq r \\ \frac{1}{a'_{rk}}(B^{-1}_{old})_{rj} & if \quad i = r \end{cases}$$

These formulas are expressed in matrix notation as

$B^{-1}_{new} = EB^{-1}_{old}$,

where matrix E is an identity matrix except that its r^{th} column is replaced by the vector

$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix}$, where

$$\eta_i = \begin{cases} \frac{-a'_{ik}}{a'_{rk}} & if \quad i \neq r \\ \frac{1}{a'_{rk}} & if \quad i = r \end{cases}$$

Let's go back and obtain B^{-1}'s in the previous example without **"any"** matrix inversion,

$\eta = \begin{bmatrix} -\frac{a_{12}}{a_{22}} \\ \frac{1}{a_{22}} \\ -\frac{a_{32}}{a_{22}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -1 \end{bmatrix}$

$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Next iteration,

$\eta = \begin{bmatrix} -\frac{a'_{11}}{a'31} \\ -\frac{a'_{21}}{a'31} \\ \frac{1}{a'31} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$

$B^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$

3.3 Fundamental Insight

After any iteration, the coefficients of the **slack** variables in each equation immediately reveal how that equation has been obtained from the initial equations. In other words, you can read B^{-1} from under the slack variables for any iteration of the simplex.

Simplex tableaux without leftmost columns for the Wyndor Glass Co. problem:

Iteration	x_1	x_2	x_3	x_4	x_5	Right Side
0	-3	-5	0	0	0	0
	1	0	**1**	**0**	**0**	4
	0	2	**0**	**1**	**0**	12
	3	2	**0**	**0**	**1**	18
1	-3	0	0	5/2	0	30
	1	0	**1**	**0**	**0**	4
	0	1	**0**	**1/2**	**0**	6
	3	0	**0**	**-1**	**1**	6
2	0	0	0	3/2	1	36
	0	0	**1**	**1/3**	**-1/3**	2
	0	1	**0**	**1/2**	**0**	6
	1	0	**0**	**-1/3**	**1/3**	2

3.4 A Note on Shadow Prices

Shadow price for resource i (denoted by y_i^*) measures the marginal value of that resource i.e, the rate at which z could be increased by (slightly) increasing the amount of this resource (b_i).

This is extremely important from a managerial standpoint considering most of the numbers in your optimization models are estimated by the operations management team.

⋆ The key question: Which constraints are binding?

In other words, which resources are depleted at optimality?

In other words, which resources are more crucial for the model?

$$
\begin{aligned}
\max\ & 3x_1 + 5x_2 \\
\text{s.t.}\ & x_1 \le 4 \quad \text{(nonbinding) at optimality} \\
& 2x_2 \le 12 \quad \text{(binding) at optimality} \\
& 3x_1 + 2x_2 \le 18 \quad \text{(binding) at optimality} \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Shadow price for second constraint is $\frac{3}{2}$. It means z increases by $\frac{3}{2}$ units if RHS value of second constraint is increased by one unit.

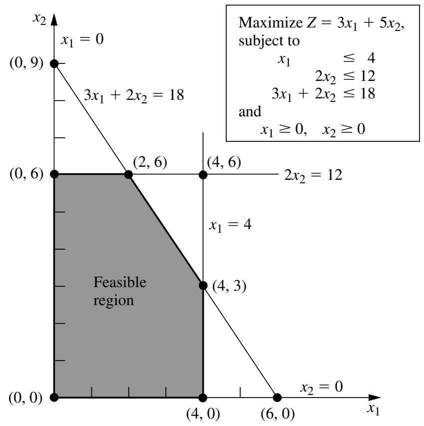
Shadow price for first constraint is 0.

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	$\boxed{0}$	$\boxed{\frac{3}{2}}$	$\boxed{1}$	36
0	0	0	1	$\frac{1}{3}$	$\frac{-1}{3}$	2
0	0	1	0	$\frac{1}{2}$	0	6
0	1	0	0	$\frac{-1}{3}$	$\frac{1}{3}$	2

Shadow price for third constraint is 1.

That optimal table and shadow prices show that second resource is relatively the most important.

Remember the graph:



These numbers also justify which constraints are binding are which ones are not.

3.5 Exercises

(1) Consider the following problem.

$$\begin{aligned} \max\ z = c_1x_1 + 2x_2 + c_3x_3 & \\ \text{s.t.}\quad a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &\le 60 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &\le 10 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &\le 20 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0. & \end{aligned}$$

Let x_4, x_5, and x_6 denote the slack variables for the first, second, and third constraints, respectively. After we apply the simplex method for a few iterations, an intermediate simplex tableau is as follows:

z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
1	0	0	0	f_{04}	3	$7/2$	r_0
0	1	0	0	1	-1	-2	r_1
0	0	1	0	$-1/2$	1	$3/2$	r_2
0	0	0	1	$3/2$	-2	$-5/2$	r_3

Suppose that the tableau above is not optimal and the optimal basis is known to be x_2, x_3, and x_4. Find the optimal solution $(z^*, x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)$.

(2) Consider the following problem.

$$\begin{aligned} \max\ 20x_1 + 6x_2 + 8x_3 & \\ \text{s.t.}\quad 8x_1 + 2x_2 + 3x_3 &\le 200 \\ 4x_1 + 3x_2 + 3x_3 &\le 100 \\ 2x_1 + 3x_2 + x_3 &\le 50 \\ x_3 &\le 20 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0. & \end{aligned}$$

Let x_4, x_5, x_6, and x_7 denote the slack variables for the first through fourth constraints, respectively.

(a) Suppose that after some number of iterations of the simplex method, x_1, x_2, x_6, and x_7 are in the basis. Is this a basic feasible solution? Explain why/why not.

(b) Starting with the basis in part (a), use revised simplex method to find the optimal solution and the optimal objective function value.

(3) Consider the following problem and **do not use simplex method** from the scratch.

$$\max \quad c_1x_1 + c_2x_2 + c_3x_3$$

$$\begin{aligned} \text{subject to } & x_1 + 2x_2 - x_3 && \leq 2 \\ & -x_1 + 4x_2 + 2x_3 && \geq 5 \\ & 3x_1 + x_2 - x_3 && \leq 4 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

(a) Suppose that after some number of iterations of the simplex method, x_1, x_2, and x_3 are in the basis. Is this a basic feasible solution? Compute values for x_1, x_2, x_3, x_4 (slack for the first constraint), x_5 (surplus for the second constraint), and x_6 (slack for the third constraint).

(b) Find values (or ranges) for c_1, c_2, and c_3 such that the solution provided in part (a) is optimal.

(c) Suppose $c_1 = 0$, $c_2 = 4$, and $c_3 = 0$. Starting with the solution in part (a), perform **one iteration** of revised simplex method.

(4) Consider the following problem.

$$\begin{aligned} \max\ & z = -2x_1 - 3x_2 - 2x_3, \\ \text{s.t. } & x_1 + 4x_2 + 2x_3 \geq 8 \\ & 3x_1 + 2x_2 + 2x_3 \geq 6 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \end{aligned}$$

Let x_4 and x_6 be the surplus variables for the first and second constraints, respectively; and $\bar{x}_5$ and $\bar{x}_7$ be the corresponding artificial variables. After you apply the simplex method, a portion of the final simplex tableau is as follows:

z	x_1	x_2	x_3	x_4	$\bar{x}_5$	x_6	$\bar{x}_7$	RHS
					$M - 0.5$		$M - 0.5$	
					0.3		-0.1	
					-0.2		0.4	

Identify the missing numbers in the above simplex tableau. Show your calculations.

(5) Solve the following problem using revised simplex method. Start with variables x_1, x_2, x_6 (slack variable for third constraint) in the basis.

$$\begin{aligned} \min\ & z = 2x_1 + 3x_2 - x_3 \\ \text{s.t. } & x_1 + 4x_2 + 2x_3 && \geq 8 \\ & 3x_1 + 2x_2 && \geq 6 \\ & x_1 + x_2 + x_3 && \leq 5 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

(6) Consider the following problem.

$$\begin{aligned} \min\ & z = 2x_1 + 3x_2 - x_3 \\ \text{s.t.}\ & x_1 + 4x_2 + 2x_3 \geq 8 \\ & 3x_1 + 2x_2 \geq 6 \\ & x_1 + x_2 + x_3 \leq 5 \\ & x_1 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

(a) Suppose that after some number of iterations of the simplex method, x_1, x_2, x_6 (slack variable for third constraint), and x_7 (slack variable for fourth constraint) are in the basis. Is this a basic feasible solution? Explain why/why not.

(b) Starting with the basis in part (a), perform one iteration of revised simplex method to identify a new solution and denote if this solution is optimal or not.

z	x_1	x_2	x_3

(7) Consider the following problem.

$$\begin{aligned} \max\ & -5x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\ \text{s.t.}\ & a_{11}x_1 + a_{12}x_2 - 3x_3 + a_{14}x_4 \geq b_1 \\ & a_{21}x_1 + a_{22}x_2 + 10x_3 + a_{24}x_4 = b_2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0. \end{aligned}$$

Suppose x_5 is the surplus variable for constraint 1 and the simplex method yields the following tableau.

z	x_1	x_2	x_3	x_4	x_5	Right Side
1	0	0	2	0	5	100
0	16	1	-2	0	-4	10
0	-1	0	3	1	1	20

Find missing values in the problem. Show all your work and write the values of missing parameters in the table below.

c_2	c_3	c_4	a_{11}	a_{12}	a_{14}	b_1	a_{21}	a_{22}	a_{24}	b_2

(8) Consider the following problem.

$$\max \quad z = 4x_1 - x_2 + 2x_3$$

$$
\begin{aligned}
\text{s.t.} \quad & 2x_1 - 2x_2 + 3x_3 \leq 5 \\
& x_1 + x_2 - x_3 \leq 3 \\
& x_1 - x_2 + x_3 \leq 2 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Let x_4, x_5, and x_6 denote the slack variables for constraints 1, 2, and 3, respectively. After you apply the simplex method, a portion of an intermediate simplex tableau is as follows:

z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
1				1	1	0	
0				1	3	0	
0				0	1	1	
0				1	2	0	

(a) Identify the missing numbers in the simplex tableau. Show your calculations.

(b) Starting with the given tableau, find the optimal solution and optimal objective function value using **revised simplex method**. **You are not allowed to invert a matrix during this procedure.**

z^*	x_1^*	x_2^*	x_3^*

3.6 Further Reading and Exercises

The reader is refereed to Chapter 5 in the textbook, *Introduction to Operations Research* by Hillier and Lieberman.

Chapter 4

Duality
(Weeks 10-11)

Each linear programming problem has a corresponding dual problem. Let's call our original linear programming problem *primal problem* and consider its *dual problem*. In this chapter, we will learn certain relationships between primal and dual problems (duality theory), that provide certain insights and help with optimization of the primal problem. These properties will also help us in the next chapter, where we conduct sensitivity analysis.

Next, we discuss how to obtain the dual formulation. Consider the following primal problem:

$$\begin{aligned} \max\ & 3000x_1 + 5000x_2 \\ \text{s.t.}\ & x_1 \le 4 \\ & 2x_2 \le 12 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Then, the corresponding dual problem is as follows:

$$\begin{aligned} \min\ & 4y_1 + 12y_2 + 18y_3 \\ \text{s.t.}\ & y_1 + 3y_3 \ge 3000 \\ & 2y_2 + 2y_3 \ge 5000 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

⋆⋆⋆ How many variables are there in the primal? In the dual?

⋆⋆⋆ How many constraints are there in the primal? In the dual?

Make sure that you digest the following correspondence between entities in primal and dual problems:

One Problem	Other problem
Constraint i	Variable i
Objective Function	Right Hand Sides

First thing to note: Dual of the dual is primal problem.

Second thing to note: You have to understand how dual is found. The following table lists the relationship between one problem and its dual.

One Problem (obj.)	**Other problem (opposite obj.)**
Constraint i	Variable i
Coefficients (in row) Constraint inequality directions	Coefficients (in column) Variable signs
Objective Function (coefficients)	Right Hand Sides (coefficients)

The inequalities and variable signs for a primal-dual pair can be summarized as follows:

max problem	**min problem**
variable ≥ 0	$\geq$ constraint
variable ≤ 0	$\leq$ constraint
variable urs.	$=$ constraint
$\geq$ constraint	variable ≤ 0
$\leq$ constraint	variable ≥ 0
$=$ constraint	variable urs.

Below is another example with all possible types of inequalities and variable signs.

Example 4.1 Construct the dual for the following problem:

$$\begin{aligned}
\max\ & 2x_1 + x_2 + 3x_3 \\
\text{s.t.}\ & x_1 + x_2 + x_3 = 3 \\
& x_1 - 2x_2 + x_3 \geq 1 \\
& 2x_2 + x_3 \leq 2 \\
& x_1 \geq 0, x_2 \leq 0, x_3 \text{ urs.}
\end{aligned}$$

Solution

$$\begin{aligned}
\min\ & 3y_1 + y_2 + 2y_3 \\
\text{s.t.}\ & y_1 + y_2 \geq 2 \\
& y_1 - 2y_2 + 2y_3 \leq 1 \\
& y_1 + y_2 + y_3 = 3 \\
& y_1 \text{urs.}, y_2 \leq 0, y_3 \geq 0
\end{aligned}$$

Example 4.2 Construct the dual for the following problem. Note that it is the same problem as above but the direction of the objective function has changed only.

$$\begin{aligned}
\min\ & 2x_1 + x_2 + 3x_3 \\
\text{s.t.}\ & x_1 + x_2 + x_3 = 3 \\
& x_1 - 2x_2 + x_3 \geq 1 \\
& 2x_2 + x_3 \leq 2 \\
& x_1 \geq 0, x_2 \leq 0, x_3 \text{ urs.}
\end{aligned}$$

Solution

$$\begin{aligned}
\max\ & 3y_1 + y_2 + 2y_3 \\
\text{s.t.}\ & y_1 + y_2 \leq 2 \\
& y_1 - 2y_2 + 2y_3 \geq 1 \\
& y_1 + y_2 + y_3 = 3 \\
& y_1 \text{urs.}, y_2 \geq 0, y_3 \leq 0
\end{aligned}$$

Matrix Representation of Primal in Canonical Form:

$$\begin{aligned}
\max\ & z = c^T x \\
\text{s.t.}\ & Ax \leq b \\
& x \geq 0
\end{aligned}$$

Matrix Representation of the Dual:

$$\begin{aligned}
\min\ & w = y^T b \\
\text{s.t.}\ & A^T y \geq c \\
& y \geq 0
\end{aligned}$$

Note that c, b, x, and y are a column vectors, and A is a matrix. Also note that cx, yb, $c_B B^{-1}$ are fairly common usages although they formally imply $c^T x$, $y^T b$, $c_B^T B^{-1}$.

4.1 Connection between the primal and dual optimal solutions

Consider the Wyndor Glass Co. Problem again.

$$\max \quad z - 3x_1 - 5x_2 + 0x_3 + 0x_4 + 0x_5 = 0$$

$$\begin{array}{lllllll}
\text{s.t.} & x_1 & & +x_3 & & & = 4 \\
& & 2x_2 & & +x_4 & & = 12 \\
& 3x_1 & +2x_2 & & & +x_5 & = 18 \\
& x_1, & x_2 & \geq 0 & & &
\end{array}$$

Primal Solution:

Iteration 0

z	x_1	x_2	x_3	x_4	x_5	RHS
1	-3	-5	0	0	0	0
0	1	0	1	0	0	4
0	0	2	0	1	0	12
0	3	2	0	0	1	18

Iteration 1

z	x_1	x_2	x_3	x_4	x_5	RHS
1	-3	0	0	$\frac{5}{2}$	0	30
0	1	0	1	0	0	4
0	0	1	0	$\frac{1}{2}$	0	6
0	3	0	0	-1	1	6

Iteration 2

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	$\frac{3}{2}$	1	36
0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
0	0	1	0	$\frac{1}{2}$	0	6
0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (2, 6, 2, 0, 0)$

$z^* = 36$

Shadow Prices for the constraints: $(0,\ \frac{3}{2},\ 1)$

Dual Problem:

$$\begin{aligned} \min\ & 4y_1 + 12y_2 + 18y_3 \\ \text{s.t.}\ & y_1 + 3y_3 \geq 3 \\ & 2y_2 + 2y_3 \geq 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

In augmented form:

$$\begin{aligned} \min\ & 4y_1 + 12y_2 + 18y_3 \\ \text{s.t.}\ & y_1 + 3y_3 - y_4 = 3 \\ & 2y_2 + 2y_3 - y_5 = 5 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0 \end{aligned}$$

Dual Solution:

Iteration 0

z	y_1	y_2	y_3	y_4	y_5	y_4'	y_5'	RHS
-1	4	12	18	0	0	M	M	0
0	1	0	3	-1	0	1	0	3
0	0	2	2	0	-1	0	1	5

Iteration 0

z	y_1	y_2	y_3	y_4	y_5	y_4'	y_5'	RHS
-1	$-M+4$	$-2M+12$	$-5M+18$	M	M	0	0	$-8M$
0	1	0	3	-1	0	1	0	3
0	0	2	2	0	-1	0	1	5

Iteration 1

z	y_1	y_2	y_3	y_4	y_5	y_4'	y_5'	RHS
-1	$\frac{2M}{3}-2$	$-2M+12$	0	$-\frac{2M}{3}+6$	M	$\frac{5M}{3}-6$	0	$-3M-18$
0	$\frac{1}{3}$	0	1	$\frac{-1}{3}$	0	$\frac{1}{3}$	0	1
0	$\frac{-2}{3}$	2	0	$\frac{2}{3}$	-1	$\frac{-2}{3}$	1	3

Iteration 2

z	y_1	y_2	y_3	y_4	y_5	y_4'	y_5'	RHS
-1	2	0	0	2	6	$M-2$	$M-6$	-36
0	$\frac{1}{3}$	0	1	$\frac{-1}{3}$	0	$\frac{1}{3}$	0	1
0	$\frac{-1}{3}$	1	0	$\frac{1}{3}$	$\frac{-1}{2}$	$\frac{-1}{3}$	$\frac{1}{2}$	$\frac{3}{2}$

Optimal Solution: $(y_1^*, y_2^*, y_3^*, y_4^*, y_5^*) = (0, \frac{3}{2}, 1, 0, 0)$
$w^* = 36$
Shadow Prices for the constraints: (2, 6)

$\star$ Can you derive any correlation between the primal and dual optimal solutions and their corresponding shadow prices?

$\star\star$ Make sure that you understand which shadow price (dual variable) indicate what. You may be able to obtain the dual optimal solution from primal optimal solution and vice versa. How?

- $c_B B^{-1}$ values in the primal solution are equal to y values of the corresponding dual solution. Think about the simplex tableau.
- Furthermore, $z^* = c_B B^{-1} b = {y^*}^T b = w^*$
  So the **optimal objective values of the primal and the dual are equal**.

For the **Economic Interpretation** of the Dual Problem, the reader is referred to *Operations Research: Applications and Algorithms* by Wayne L. Winston. In Section 6.6 of that book, the dual of the Dakota problem (introduced in Section 4.5) is discussed in detail.

4.2 Summary of Primal-Dual Relationships

Weak duality property: If x is a feasible solution for the maximization problem and y is a feasible solution for the corresponding dual minimization problem, then $cx \leq yb$. You can observe this from each iteration of the simplex algorithm for both primal and dual.
Strong duality property: If x^* is an optimal solution for the primal problem and y^* is an optimal solution for the dual problem, then $cx^* = y^*b$.
Complementary solutions property: At each iteration, the simplex method simultaneously identifies a CPF solution x for the primal problem and a complementary solution y for the dual problem (found in row 0, the coefficients of the slack variables), where $cx = yb$.

If x is not optimal for the primal problem, then y is not feasible for the dual problem.

For a primal problem of size m by n, i.e., m constraints and n original variables, the following relationships between complementary basic solutions (also known as complementary slackness) hold:

Primal Variable	Associated Dual Variable	Number of Variables
Basic	Nonbasic	m
Nonbasic	Basic	n

The key here is to understand which dual variable, each variable is associated with. Let's revisit our previous examples and understand that association, which is extremely important.

$$\begin{aligned} \textbf{[P1]} \max\ & 3x_1 + 5x_2 \\ \text{s.t. } & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \textbf{[D1]} \min\ & 4y_1 + 12y_2 + 18y_3 \\ \text{s.t. } & y_1 + 3y_3 \geq 3 \\ & 2y_2 + 2y_3 \geq 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Solution & Obj. Func. Val.	**Solution & Obj. Func. Val.**

Note that you need to work out with the original problem to find the associated dual. Next, you will need to write both primal and dual in standard form to observe a full picture of all variables.

$$
\begin{aligned}
\textbf{[P2]}\ \max\ & 2x_1 + x_2 + 3x_3 \\
\text{s.t. } & x_1 + x_2 + x_3 = 3 \\
& x_1 - 2x_2 + x_3 \geq 1 \\
& 2x_2 + x_3 \leq 2 \\
& x_1 \geq 0, x_2 \leq 0, x_3 \text{ urs.}
\end{aligned}
\qquad
\begin{aligned}
\textbf{[D2]}\ \min\ & 3y_1 + y_2 + 2y_3 \\
\text{s.t. } & y_1 + y_2 \geq 2 \\
& y_1 - 2y_2 + 2y_3 \leq 1 \\
& y_1 + y_2 + y_3 = 3 \\
& y_1 \text{urs.}, y_2 \leq 0, y_3 \geq 0
\end{aligned}
$$

Solution & Obj. Func. Val.	**Solution & Obj. Func. Val.**

An unrestricted variable is always basic, even if it is equal to zero!

Complementary optimal solutions property: At the final iteration, the simplex method simultaneously identifies an optimal solution x^* for the primal problem and a complementary optimal solution y^* for the dual problem (found in row 0, the coefficients of the slack variables), where $cx^* = y^*b$. The y_i^*'s are the *shadow prices* for the primal problem.

$\star\star\star$ Make sure that you see each corresponding dual (infeasible) solution for the steps of simplex algorithm on primal problem in Section 4.1.

$\star$ In general, you can see corresponding primal (infeasible) solutions for the steps of simplex algorithm on the dual problem. However, in this case in Section 4.1, when a feasible solution is found for the dual (whose objective is not a function of M), we immediately reach optimality so we only see a dual and primal feasible solution, thus optimality.

$\star$ Note that dual feasibility in simplex is synonymous to optimality conditions.

4.3 Duality Theory

- If one problem (primal or dual) is feasible and bounded so is the other.
- If one problem (primal or dual) is feasible but unbounded than the other is infeasible.
- If one problem (primal or dual) is infeasible then the other is either infeasible or unbounded.

$\star$ Think about the cases of degeneracy and alternative optima. What will be observed in the dual?

Notes on the Simplex Method

- The simplex method finds a specific pair of solutions (x, y) for the primal and dual problems at each iteration such that $cx = yb$
- The primal solution is feasible but the dual solution is not feasible except for the optimal solution.
- When the primal reaches optimality, the dual solution found becomes feasible, which is also optimal.

Notes on complementary slackness

In a CP solution to an LP problem, when a dual (primal) variable is basic then the slack (surplus) variable in the corresponding primal (dual) constraint is nonbasic.
In iteration 2 of Wyndor Glass Co. Problem, primal basic variables: x_1, x_2, x_3 and corresponding nonbasic dual variables: y_4, y_5, y_1

4.4 Exercises

(1) Find the dual of the following problem.

$$\begin{aligned} \min \quad & 2x_1 + 15x_2 + 5x_3 + 6x_4 \\ \text{s.t.} \quad & -2x_1 + 5x_2 - 4x_3 + 3x_4 \leq -3 \\ & x_1 + 6x_2 + 3x_3 + x_4 \geq 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(2) Consider the following problem:

$$\begin{aligned} \max \ & 4x_1 + x_2 \\ \text{s.t.} \ & x_1 + x_2 \leq 10 \\ & 3x_1 + x_2 \geq 5 \\ & x_1 \leq 0, x_2 \geq 0 \end{aligned}$$

(a) Construct the corresponding dual problem.
(b) Find all corner point solutions for the primal problem and their associated dual solutions using *complementary slackness*.

(3) Consider the following problem.

$$\begin{aligned} \min \quad & 2x_1 + 2x_3 - x_4 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 \leq 8 \\ & 2x_1 - x_2 + 3x_3 - 2x_4 \geq 5 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

a. Construct the optimal tableau.
b. Would the optimal basis stay feasible when b_1 changes from 8 to 1?

(4) Give the dual of the following problem. Identify corner point primal and corresponding corner point dual solutions.

$$\begin{aligned} \max \quad & -2x_1 + 3x_2 + 5x_3 \\ \text{s.t.} \quad & -2x_1 + x_2 + 3x_3 + x_4 \geq 5 \\ & 2x_1 + x_3 = 4 \\ & -2x_2 + x_3 + x_4 \leq 6 \\ & x_1 \leq 0 \\ & x_2, x_3 \geq 0 \end{aligned}$$

(5) Consider the following problem.

$$\begin{aligned} \min\ & c_2x_2 - x_1 - x_3 \\ \text{s.t.}\ & x_3 - x_1 - x_2 \leq 1 \\ & 2x_1 - x_2 + x_3 \leq 2 \\ & -2 \leq x_1 \leq 1 \\ & -2 \leq x_3 \leq -1 \end{aligned}$$

(a) Construct the dual problem.

(b) Use duality theory and explain if $(x_1, x_2, x_3) = (-1, 2, -1)$ can be an optimal solution. Does your answer depend on c_2? If so, provide necessary conditions.

(c) Use duality theory and explain if $(x_1, x_2, x_3) = (1/3, -10/3, -2)$ can be an optimal solution. Does your answer depend on c_2? If so, provide necessary conditions.

(d) Provide a feasible corner point solution (x_1, x_2, x_3) other than those provided above and obtain the corresponding dual solution.

(6) Suppose we have a problem P and its dual D. If P is infeasible and there exists a feasible solution for D, then what can we conclude about D? Explain.

4.5 Further Reading and Exercises

The reader is refereed to Chapter 6 (up to Section 6.5) in the textbook, *Introduction to Operations Research* by Hillier and Lieberman.

Chapter 5

Sensitivity Analysis (Weeks 12-14)

We are not done when the simplex method has been successfully applied to identify an optimal solution for the model. It is important to perform **sensitivity analysis** to investigate the effect on the optimal solution provided by the simplex method if the parameters take on other possible values. Usually there will be some parameters that can be assigned any reasonable value without the optimality of this solution being affected. However, there may also be parameters with likely alternative values that would yield a new optimal solution. This situation is particularly serious if the original solution would then have a substantially inferior value of the objective function, or perhaps even be infeasible!

Therefore, one main purpose of sensitivity analysis is to identify the sensitive parameters (i.e., the parameters whose values cannot be changed without changing the optimal solution). For certain parameters that are not categorized as sensitive, it is also very helpful to determine the range of values of the parameter over which the optimal solution will remain unchanged. (*We call this range of values the allowable range to stay optimal.*) In some cases, changing a parameter value can affect the feasibility of the optimal BF solution. For such parameters, it is useful to determine the range of values over which the optimal BF solution (with adjusted values for the basic variables) will remain feasible. (*We call this range of values the allowable range to stay feasible.*)

GOOD NEWS: We won't learn something new (except for the dual simplex method) in this chapter! We will simply be *applying* the theory of simplex method and insights of revised simplex method in a new context.

5.1 Changes in the Right Hand Side

What would happen to the optimal objective function value of this problem if the RHS of this constraint was increased by one?

The shadow price of that constraint is the answer to this question. Remember our discussion on the interpretation of shadow prices.

Example 5.1 Remember the Wyndor Glass Co. Example given as follows:

$$\begin{aligned} \max\ & z = 3x_1 + 5x_2 \\ \text{s.t.}\ & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Final tableau is as follows:

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	$\frac{3}{2}$	1	36
0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
0	0	1	0	$\frac{1}{2}$	0	6
0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Optimal solution: $x_1^* = 2$, $x_2^* = 6$, $x_3^* = 2$, $z^* = 36$

- How will the optimal solution be affected if the RHS of the second constraint is increased by one unit?
  $2x_2 \leq 13$
  The optimal solution changes as much as shadow price value of second constraint.
  $z_{new}^* = 36 + y_2 = 36 + [c_B B^{-1}]_2 = 36 + \frac{3}{2}$

- Assume that b has been changed as b'.
  Calculate the new RHS values in the final tableau as:
  $x_B^* = B^{-1}b'$
  Calculate the new z^* as:
  $z^* = c_B B^{-1}b'$

Example 5.2 Consider Wyndor Glass Co. Example again. Assume that RHS value of third constraint has increased to 30.

$3x_1 + 2x_2 \leq 30$

$$b' = \begin{bmatrix} 4 \\ 12 \\ 30 \end{bmatrix} \quad B^{-1}b' = \begin{bmatrix} 1 & \frac{1}{3} & \frac{-1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{-1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 30 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 6 \end{bmatrix}$$

This indicates that the current solution will no longer be feasible.

In this case, if we want to find the new solution, we need to apply *dual simplex* (will be discussed later), to find a primal feasible solution.

Range for Feasibility for b

Example 5.3 What is the range for feasibility for b_3 which is 18 in the original problem?

$$B^{-1}b' = \begin{bmatrix} 1 & \frac{1}{3} & \frac{-1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{-1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18+\Delta \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2 + \frac{-\Delta}{3} \geq 0 \quad \Rightarrow \quad \Delta \leq 6$

$6 \geq 0$

$2 + \frac{\Delta}{3} \geq 0 \quad \Rightarrow \quad \Delta \geq -6$

$-6 \leq \Delta \leq 6 \quad \Rightarrow \quad 12 \leq b_3 \leq 24$

As long as we stay in this range, the optimal solution and the shadow price for the third constraint will not change.

5.2 Changes in the Coefficients of a Nonbasic Variable

Changes in the coefficient of a nonbasic variable in objective function may change the basis and the optimal solution.

Example 5.4 Consider the following new problem (this is not Wyndow Glass Co. example)

$$\begin{aligned} \max\ & 3x_1 + 5x_2 + 4x_3 \\ \text{s.t.}\ & x_1 \leq 4 \\ & 2x_2 + 2x_3 \leq 12 \\ & 3x_1 + 2x_2 + x_3 \leq 18 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Final Tableau

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	0	0	$\frac{3}{2}$	1	36
0	0	$\frac{-1}{3}$	0	1	$\frac{1}{6}$	$\frac{-1}{3}$	0
0	0	1	1	0	$\frac{1}{2}$	0	6
0	1	$\frac{1}{3}$	0	0	$\frac{-1}{6}$	$\frac{1}{3}$	4

Assume that the coefficient of second variable in the objective function has increased to 8. We should check optimality of the current basis. Why don't we need to check feasibility?

$$
\begin{aligned}
c_B B^{-1} N - c_N &= [0\ 4\ 3] \begin{bmatrix} \frac{-1}{3} & \frac{1}{6} & \frac{-1}{3} \\ 1 & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{-1}{6} & \frac{1}{3} \end{bmatrix} - [8\ 0\ 0] \\
&= [5, \frac{3}{2}, 1] - [8\ 0\ 0] \\
&= [-3, \frac{3}{2}, 1]
\end{aligned}
$$

This is not optimal. We should continue with next simplex iteration.

5.3 Changes in the Coefficients of a Basic Variable

Example 5.5 Consider Wyndor Glass Co. Example again.

Assume that coefficient of x_1 in the objective function has decreased from 3 to 1. Final tableau of the original problem:

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	$\frac{3}{2}$	1	36
0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
0	0	1	0	$\frac{1}{2}$	0	6
0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

$c_B B^{-1} A_1 - c_1' = c_B B^{-1} A_1 - (c_1 - 2) = 0 + 2 = 2$. Thus, we obtain

z	x_1	x_2	x_3	x_4	x_5	RHS
1	2	0	0	$\frac{3}{2}$	1	36
0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
0	0	1	0	$\frac{1}{2}$	0	6
0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Note that this tableau is not in the proper form. We apply simple row operations and obtain

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	$\frac{13}{6}$	$\frac{1}{3}$	32
0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
0	0	1	0	$\frac{1}{2}$	0	6
0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

We are done as this final tableau is optimal! If this was not optimal, we would proceed with regular simplex iterations. Note that same result would be obtained if we updated $c'_B B^{-1}N - c_N$ and $c'_B B^{-1}b$ directly as this is the only part that would change with a change in c_B. Make sure that you practice that in this problem.

$\star$ How would you compute the range for optimality for a change in objective function coefficient of a basic variable?

$\star$ Can we make the current basis feasible but not optimal with a change in objective function coefficient of a basic variable?

$\star\star\star$ Can we make the current basis infeasible with a change in objective function coefficient of a basic variable?

$\star\star\star$ What will happen with a change in constraint coefficient of a basic (or nonbasic) variable?

$\star\star\star$ We will not cover *adding a new variable* explicitly but you are responsible for that. It is quite straightforward - you update the corresponding column for the appropriate simplex tableau and check if that new variable enters the basis or not. Make sure that you solve the exercises at the end of this chapter.

Example 5.6 Solve the following problem in class.

$$\begin{aligned}
\max\ & z = -5x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{s.t.}\ & a_{11}x_1 + a_{12}x_2 - 3x_3 + a_{14}x_4 \quad \geq b_1 \\
& a_{21}x_1 + a_{22}x_2 + 10x_3 + a_{24}x_4 \quad = b_2 \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.
\end{aligned}$$

Suppose x_5 is the surplus variable for constraint 1 and the simplex method yields the following final tableau.

z	x_1	x_2	x_3	x_4	x_5	Right Side
1	0	0	2	0	5	100
0	16	1	-2	0	-4	10
0	-1	0	3	1	1	20

(1) Find the right side of the original problem $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and allowable range to stay feasible for b_1.

⋆⋆⋆ How would you compute B^{-1}?

$$\boxed{B^{-1} = \begin{bmatrix} 4 & a \\ -1 & b \end{bmatrix} \text{ using the surplus column, i.e., } B^{-1}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}}$$

$B^{-1}A_3 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 4 & a \\ -1 & b \end{bmatrix} \begin{bmatrix} -3 \\ 10 \end{bmatrix} = \begin{bmatrix} -12 + 10a \\ 3 + 10b \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$-12a + 10a = -2$

$3 + 10b = 3$

$a = 1, b = 0$

$B^{-1} = \begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix}$

$B^{-1}b = \begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

$b_1 = -20, b_2 = 90$

Allowable range to stay feasible for b_1:

$B^{-1}b' = \begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} b_1 + \Delta \\ b_2 \end{bmatrix} = \begin{bmatrix} 10 + 4\Delta \\ 20 - \Delta \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2.5 \leq \Delta \leq 20 \Rightarrow -22.5 \leq b_1 \leq 0$

(2) Find the coefficient of x_4 in the objective function of the original problem (i.e., c_4) and allowable range to stay optimal for c_4.

Note that x_4 is a basic variable!

$c_B B^{-1} N - c_N = [c_2 \quad c_4] \begin{bmatrix} 16 & -2 & -4 \\ -1 & 3 & 1 \end{bmatrix} - [-5 \quad c_3 \quad 0] = [0 \quad 2 \quad 5]$

$[16c_2 - c_4 + 5 \quad -2c_2 + 3c_4 - c_3 \quad -4c_2 + c_4] = [0 \quad 2 \quad 5]$

$c_2 = 0 \quad c_3 = 13 \quad c_4 = 5$

$[0 \quad 5 + \Delta] \begin{bmatrix} 16 & -2 & -4 \\ -1 & 3 & 1 \end{bmatrix} - [-5 \quad 13 \quad 0] \geq 0$

$[-\Delta \quad 2 + 3\Delta \quad 5 + \Delta] \geq 0$

$\Delta \leq 0 \qquad \Delta \geq -2/3 \qquad \Delta \geq -5$

$-2/3 \leq \Delta \leq 0 \Rightarrow 13/3 \leq c_4 \leq 5$

(3) Find the coefficient of x_3 in the objective function of the original problem (i.e., c_3) and

allowable range to stay optimal for c_3.

Note that x_3 is a nonbasic variable!

$c_3 = 13$

$c_B B^{-1} N - c_N = [0 \quad 2-\Delta \quad 5] \geq 0$

$2 - \Delta \geq 0 \qquad \Delta \leq 2 \Rightarrow c_3 \leq 15$

5.4 Dual Simplex Method

⋆ Think about the cases where **RHS is negative but the row 0 of tableau (objective function row) looks optimal**. That's when Dual Simplex Method is useful because this is the case where dual problem is feasible but primal is not!

⋆ Which of the following are possible candidates for using Simplex Method or Dual Simplex Method?

- Change in b
- Change in c_B
- Change in c_N
- Change in A_j where $j \in B$
- Change in A_j where $j \in N$

Summary of the Dual Simplex Method

Initialization: Find a basic solution such that the coefficients in row 0 are zero for basic variables and nonnegative for nonbasic variables (so the solution is optimal if it is feasible). Go to the feasibility test.

Feasibility test: Check to see whether all the basic variables are nonnegative. If they are, then this solution is feasible, and therefore optimal, so stop. Otherwise, go to an iteration.

Iteration: *Step 1:* Determine the leaving basic variable: Select the negative basic variable that has the largest absolute value.

Step 2: Determine the entering basic variable: Select the nonbasic variable whose coefficient in row 0 reaches zero first as an increasing multiple of the equation containing the leaving basic variable is added to row 0. This selection is made by checking the nonbasic variables with negative coefficients in that equation (the one containing the leaving basic variable) and selecting the one with **the smallest absolute value of the ratio of the row 0 coefficient to the coefficient in that equation**.

Step 3: Determine the new basic solution: Starting from the current set of equations, solve for the basic variables in terms of the nonbasic variables by Gaussian elimination.

When we set the nonbasic variables equal to zero, each basic variable (and z) equals the new right-hand side of the one equation in which it appears (with a coefficient of +1). Return to the feasibility test.

Example 5.7 Solve the following in class: Apply dual simplex method to the dual of Wyndor Glass Co. example that is formulated as

$$\begin{aligned} \min\ & 4y_1 + 12y_2 + 18y_3 \\ \text{s.t.}\ & y_1 + 3y_3 \geq 3 \\ & 2y_2 + 2y_3 \geq 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Solution

z	y_1	y_2	y_3	y_4	y_5	RHS
-1	4	12	18	0	0	0
0	-1	0	-3	1	0	-3
0	0	$\boxed{-2}$	-2	0	1	-5

z	y_1	y_2	y_3	y_4	y_5	RHS
-1	4	0	6	0	6	-30
0	-1	0	$\boxed{-3}$	1	0	-3
0	0	1	1	0	-1/2	5/2

z	y_1	y_2	y_3	y_4	y_5	RHS
-1	2	0	0	2	6	-36
0	1/3	0	1	-1/3	0	1
0	-1/3	1	0	1/3	-1/2	3/2

Example 5.8 Suppose that we introduce a new constraint $2x_1 + x_3 + 3x_4 \leq 50$ to the problem in Example 5.6. Find the optimal solution.

Solution

$2x_1 + x_3 + 3x_4 + x_6 = 50$ can be added to the final simplex tableau.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	2	0	5	0	100
0	16	1	-2	0	-4	0	10
0	-1	0	3	1	1	0	20
0	2	0	1	3	0	1	50
1	0	0	2	0	5	0	100
0	16	1	-2	0	-4	0	10
0	-1	0	3	1	1	0	20
0	5	0	$\boxed{-8}$	0	-3	1	-10
1	1.25	0	0	0	4.25	0.25	97.5
0	14.75	1	0	0	-3.25	-0.25	12.5
0	0.875	0	0	1	-0.125	0.375	16.25
0	-0.625	0	1	0	0.375	-0.125	1.25

⋆⋆⋆ How would you solve it if it was an equal to constraint?

ANSWER: Try your luck with another variable to be basic or (ideally) introduce an artificial variable and use two-phase/Big-M.

⋆⋆⋆ Pay attention to how tableau is updated in two different ways when a new variable is added versus a new constraint is added!

5.5 Exercises

(1) Consider the following problem.

$$\text{Max} \quad Z = -5x_1 + 5x_2 + 13x_3$$

subject to

$$-x_1 + x_2 + 3x_3 \leq 20$$
$$12x_1 + 4x_2 + 10x_3 \leq 90$$

and

$$x_j \geq 0 \quad (j = 1, 2, 3).$$

If we let x_4 and x_5 be the slack variables for the respective constraints, the simplex method yields the following final set of equations:

$$\begin{array}{llllll} (0)\ Z & & & +2x_3 & +5x_4 & = 100 \\ (1) & -x_1 & +x_2 & +3x_3 & +x_4 & = 20 \end{array}$$

$$(2) \quad 16x_1 \qquad -2x_3 \; -4x_4 \; +x_5 = 10.$$

Now you are to conduct sensitivity analysis by *independently* investigating each of the following nine changes in the original model. For each change, use the sensitivity analysis procedure to revise this set of equations (in tableau form) and convert it to proper form from Gausian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. If either test fails, reoptimize to find a new optimal solution.

(a) Change the right-hand side of the constraint 1 to $b_1 = 30$.

(b) Change the right-hand side of the constraint 2 to $b_1 = 70$.

(c) Change the right-hand sides to

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 100 \end{bmatrix}$$

(d) Change the coefficient of x_3 in the objective function to $c_3 = 8$.

(e) Change the coefficients of x_1 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

(f) Change the coefficients of x_2 to

$$\begin{bmatrix} c_2 \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

(g) Introduce a new variable x_6 with coefficients

$$\begin{bmatrix} c_6 \\ a_{16} \\ a_{26} \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix}$$

(h) Introduce a new constraint $2x_1 + 3x_2 + 5x_3 \leq 50$ (Denote its slack variable by x_6.)

(i) Change constraint 2 to $10x_1 + 5x_2 + 10x_3 \leq 100$.

(2) Consider the following problem.

$$\text{Max} \quad Z = \quad 2x_1 - x_2 + x_3$$

subject to

$$3x_1 + x_2 + x_3 \leq 60$$
$$x_1 - x_2 + 2x_3 \leq 10$$
$$x_1 + x_2 - x_3 \leq 20$$

and

$$x_j \geq 0 \quad (j = 1, 2, 3).$$

Let x_4, x_5, and x_6 denote the slack variables for the respective constraints. After we apply the simplex method, the final simplex tableau is

Basic Variables	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1	0	0	3/2	0	3/2	1/2	25
x_4	(1)	0	0	0	1	1	-1	-2	10
x_1	(2)	0	1	0	1/2	0	1/2	1/2	15
x_2	(3)	0	0	1	-3/2	0	-1/2	1/2	5

Now you are to conduct sensitivity analysis by *independently* investigating each of the following six changes in the original model. For each change, use the sensitivity analysis procedure to revise this final tableau and convert it to proper form from Gaussian elimination for identifying and evaluating the curent basic solution. Then test this solution for feasibility and for optimality. If either test fails, reoptimize to find a new optimal solution.

(a) Change the right-hand sides

$$\text{from} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix} \text{ to } \begin{bmatrix} b_6 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 70 \\ 20 \\ 10 \end{bmatrix}$$

(b) Change the coefficients of x_1

$$\text{from} \begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix} \text{ to } \begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

(c) Change the coefficients of x_3

$$\text{from} \begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix} \text{ to } \begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}$$

(d) Change the objective function to $Z = 3x_1 - 2x_2 + 3x_3$.

(e) Introduce a new constraint $3x_1 - 2x_2 + x_3 \leq 30$. (Denote its slack variable by x_7.)

(f) Introduce a new variable x_8 with coefficients

$$\begin{bmatrix} c_8 \\ a_{18} \\ a_{28} \\ a_{38} \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 2 \end{bmatrix}$$

(3) Consider the following problem.

$$\begin{aligned} \max \quad & Z = 2x_1 + 7x_2 - 3x_3 \\ \text{subject to} & \\ & x_1 + 3x_2 + 4x_3 \leq 30 \\ & x_1 + 4x_2 - x_3 \leq 10 \\ \text{and} & \\ & x_j \geq 0 \quad (j = 1, 2, 3). \end{aligned}$$

By letting x_4 and x_5 be the slack variables for the respective constraints, the simplex tableau yields the following final set of equations:

$$\begin{array}{llllllll} (0) & Z & +x_2 & +x_3 & & +2x_5 & = 20 \\ (1) & & -x_2 & +5x_3 & +x_4 & -x_5 & = 20 \\ (2) & x_1 & +4x_2 & -x_3 & & +x_5 & = 10 \end{array}$$

Now you are to conduct sensitivity analysis by *independently* investigating each of the following seven changes in the original model. For each change, use the sensitivity analysis procedure to revise this set of equations (in tableau form) and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. If either test fails, reoptimize to find a new optimal solution.

(a) Change the right-hand sides to

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

(b) Change the coefficients of x_3 to

$$\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

(c) Change the coefficients of x_1 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

(d) Introduce a new variable x_6 with coefficients

$$\begin{bmatrix} c_6 \\ a_{16} \\ a_{26} \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

(e) Change the objective function to $Z = x_1 + 5x_2 - 2x_3$.

(f) Introduce a new constraint $3x_1 + 2x_2 + 3x_3 \leq 25$.

(g) Change the constraint 2 to $x_1 + 2x_2 + 2x_3 \leq 35$.

(4) Use the dual simplex method manually to solve the following problem.

$$\begin{aligned} \min \quad Z = \quad & 7x_1 + 2x_2 + 5x_3 + 4x_4 \\ \text{subject to} \quad & \\ & 2x_1 + 4x_2 + 7x_3 + x_4 \geq 5 \\ & 8x_1 + 4x_2 + 6x_3 + 4x_4 \geq 8 \\ & 3x_1 + 8x_2 + x_3 + 4x_4 \geq 4 \\ & x_j \geq 0 \quad (j = 1, 2, 3, 4). \end{aligned}$$

(5) Solve the following problem.

$$\begin{aligned} \min \quad Z = \quad & 5x_1 + 3x_2 + 5x_3 + 4x_5 \\ \text{subject to} \quad & \\ & x_1 + x_3 \leq 5 \end{aligned}$$

$$x_2 + x_5 \geq 3$$
$$x_1 + x_2 + x_3 + x_4 = 7$$
$$x_j \geq 0 \quad (j = 1, 2, 3, 4, 5).$$

5.6 Further Reading and Exercises

The reader is refereed to Chapters 6 (after Section 6.5) and 7 in the textbook, *Introduction to Operations Research* by Hillier and Lieberman.

Chapter 6

Solutions for Exercises

The answers provided here are for reference only and are not necessarily complete/correct. Please ask your questions to the instructor or TA.

6.1 Chapter 1 Exercises

(6) <u>Decision variables:</u>

x_{11}: Amount of raw material 1 used in product 1

x_{12}: Amount of raw material 1 used in product 2

x_{21}: Amount of raw material 2 used in product 1

x_{22}: Amount of raw material 2 used in product 2

<u>Model that minimizes cost:</u>

$$\begin{array}{ll}
\min & z = 2(x_{11}+x_{12}+x_{21}+x_{22}) + (x_{11}+x_{12}) + 1.5(x_{21}+x_{22}) \\
\text{s.t.} & \quad x_{11}+x_{21} \geq 200 \\
& \quad x_{12}+x_{22} \geq 400 \\
& \quad x_{11}+x_{12} \geq 150 \\
& \quad x_{21}+x_{22} \geq 150 \\
& \quad 0.03x_{11}+0.02x_{21} \leq 0.022(x_{11}+x_{21}) \\
& \quad 0.03x_{12}+0.02x_{22} \leq 0.025(x_{11}+x_{21}) \\
& \quad x_{11}+x_{12} \leq 0.75(x_{11}+x_{12}+x_{21}+x_{22}) \\
& \quad x_{21}+x_{22} \leq 0.75(x_{11}+x_{12}+x_{21}+x_{22}) \\
& \quad x_{11}, x_{12}, x_{21}, x_{22} \geq 0
\end{array}$$

Model that maximizes profit:

$$\begin{array}{ll}
\max & z = 10(x_{11}+x_{21}) + 8(x_{12}+x_{22}) - 2(x_{11}+x_{12}+x_{21}+x_{22}) - (x_{11}+x_{12}) - 1.5(x_{21}+x_{22}) \\
\text{s.t.} & x_{11}+x_{21} \geq 200 \\
 & x_{12}+x_{22} \geq 400 \\
 & x_{11}+x_{12} \geq 150 \\
 & x_{21}+x_{22} \geq 150 \\
 & 0.03x_{11}+0.02x_{21} \leq 0.022(x_{11}+x_{21}) \\
 & 0.03x_{12}+0.02x_{22} \leq 0.025(x_{12}+x_{22}) \\
 & x_{11}+x_{12} \leq 0.75(x_{11}+x_{12}+x_{21}+x_{22}) \\
 & x_{21}+x_{22} \leq 0.75(x_{11}+x_{12}+x_{21}+x_{22}) \\
 & x_{11}, x_{12}, x_{21}, x_{22} \geq 0
\end{array}$$

(7) Decision variables:

x_1: Number of notebook computers to be produced

x_2: Number of desktop computers to be produced

x_3: Number of tablets to be produced.

Model:

$$\begin{array}{ll}
\max & z = 600x_1 + 700x_2 + 500x_3 \\
\text{s.t.} & x_1 + x_2 + x_3 \leq 15,000 \\
 & 2x_1 + 4x_3 + x_3 \leq 25,000 \\
 & 5x_1 + 4x_2 + 3x_3 \leq 25,000 \\
 & x_1, x_2, x_3 \geq 0
\end{array}$$

(8) Decision variables:

x_1:Amount of gas transferred through link 1

x_2:Amount of gas transferred through link 2

x_3:Amount of gas transferred through link 3

x_4:Amount of gas transferred through link 4

x_5:Amount of gas transferred through link 5

x_6:Amount of gas transferred through link 6.

Model:

$$
\begin{aligned}
\min \quad & z = 20x_1 + 25x_2 + 10x_3 + 15x_4 + 20x_5 + 40x_6 \\
\text{s.t.} \quad & x_1 + x_2 = 1200 \\
& x_5 + x_6 = 1200 \\
& x_1 \leq 500 \\
& x_2 \leq 900 \\
& x_3 \leq 700 \\
& x_4 \leq 400 \\
& x_5 \leq 600 \\
& x_6 \leq 1000 \\
& x_1 + x_4 = x_3 + x_5 \\
& x_2 + x_3 = x_4 + x_6 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

(9) <u>Decision variables:</u>

x_1:Number of special risk insurance

x_2:Number of mortgage.

<u>Model:</u>

$$
\begin{aligned}
\max \quad & z = 5x_1 + 2x_2 \\
\text{s.t.} \quad & 3x_1 + 2x_2 \leq 2400 \\
& 2x_2 \leq 800 \\
& 2x_1 \leq 1200 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

(10) <u>Decision variables:</u>

x_1:Number of hot dogs to be produced

x_2:Number of hot dog buns to be produced.

<u>Model:</u>

$$
\begin{aligned}
\max \quad & z = 0.2x_1 + 0.1x_2 \\
\text{s.t.} \quad & 0.1x_2 \leq 200 \\
& 0.25x_1 \leq 800 \\
& 3x_1 + 2x_2 \leq 12000
\end{aligned}
$$

$$x_1, x_2 \geq 0$$

(11) Decision variables:

x_{ij}: Amount of space leased in month i for j months, $i, j = 1, 2, 3, 4, 5$.

Model:

$$\begin{aligned}
\min \quad & z = 65\sum_{i=1}^{5} x_{i1} + 100\sum_{i=1}^{4} x_{i2} + 135\sum_{i=1}^{3} x_{i3} + 160\sum_{i=1}^{2} x_{i4} + 190x_{15} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \geq 30,000 \\
& x_{12} + x_{13} + x_{14} + x_{15} + x_{21} + x_{22} + x_{23} + x_{24} \geq 20,000 \\
& x_{13} + x_{14} + x_{15} + x_{22} + x_{23} + x_{24} + x_{31} + x_{32} + x_{33} \geq 40,000 \\
& x_{14} + x_{15} + x_{23} + x_{24} + x_{32} + x_{33} + x_{41} + x_{42} \geq 10,000 \\
& x_{15} + x_{24} + x_{33} + x_{42} + x_{51} \geq 50,000 \\
& x_{ij} \geq 0, \quad \forall i, j \quad i, j = 1, 2, 3, 4, 5.
\end{aligned}$$

Note: $x_{25}, x_{34}, x_{35}, x_{43}, x_{44}, x_{45}, x_{52}, x_{53}, x_{54}, x_{55}$ are introduced but never used above, so they can be anything, which won't effect the solution. $x_{25} = x_{34} = x_{35} = x_{43} = x_{44} = x_{45} = x_{52} = x_{53} = x_{54} = x_{55} = 0$ can be (optionally) added.

6.2 Chapter 2 Exercises

First Set:

(1)

- $x_1 = 0, x_2 = 0$, feasible
- $x_1 = 4, x_2 = 0$, feasible
- $x_1 = 6, x_2 = 0$, infeasible
- $x_1 = 4, x_2 = 3$, feasible
- $x_1 = 4, x_2 = 6$, infeasible
- $x_1 = 2, x_2 = 6$, feasible
- $x_1 = 0, x_2 = 6$, feasible

(2) $x_1, x_2, x_3, x_4, x_5 =$

- (0,0,0,1,2), feasible
- (0,0,1/2,0,0), feasible
- (0,1,0,0,2), feasible
- (1,0,0,0,1), feasible
- (2,0,0,-1,0), infeasible
- (2,-1,0,0,0), infeasible

(3) b. $x_1, x_2, x_3, x_4 =$

- (0,0,6,6), $x_1 \geq 0, x_2 \geq 0$
- (0,3,3,0), $x_1 \geq 0, x_1 + 2x_2 \leq 6$
- (3,0,0,3), $x_2 \geq 0, 2x_1 + x_2 \leq 6$
- (2,2,0,0), $2x_1 + x_2 \leq 6, x_1 + 2x_2 \leq 6$

c.

- (0,0,6,6): (0,3,3,0), (3,0,0,3)
- (0,3,3,0): (0,0,6,6), (2,2,0,0)
- (3,0,0,3): (0,0,6,6), (2,2,0,0)
- (2,2,0,0): (0,3,3,0), (3,0,0,3)

d.

- (0,0,6,6): $z = 0$
- (0,3,3,0): $z = 6$
- (3,0,0,3): $z = 9$

- (2,2,0,0): $z = 10$

(4)

x_1	x_2	x_3	x_4	x_5	x_6	RHS
-1	-2	-4	0	0	0	0
3	1	5	1	0	0	10
1	4	1	0	1	0	8
2	0	2	0	0	1	7

x_1	x_2	x_3	x_4	x_5	x_6	RHS
1 2/5	-1 1/5	0	4/5	0	0	8
3/5	1/5	1	1/5	0	0	2
2/5	3 4/5	0	- 1/5	1	0	6
4/5	- 2/5	0	- 2/5	0	1	3

x_1	x_2	x_3	x_4	x_5	x_6	RHS
1 10/19	0	0	14/19	6/19	0	9 17/19
11/19	0	1	4/19	- 1/19	0	1 13/19
2/19	1	0	- 1/19	5/19	0	1 11/19
16/19	0	0	- 8/19	2/19	1	3 12/19

Optimal Solution (x_1^*, x_2^*, x_3^*)=(0, 1 11/19,1 4/7) and $z^* = -9 - 17/19$.

Second Set:

(1)

$$\begin{aligned}
\max \quad & z = -2x_1 - 3x_2 - x_3 - Mx'_4 - Mx'_5 \\
\text{s.t.} \quad & x_1 + 4x_2 + 2x_3 - x_4 + x'_4 = 8 \\
& 3x_1 + 2x_2 - x_5 + x'_5 = 6 \\
& x_1, x_2, x_3, x_4, x'_4, x_5, x'_5 \geq 0
\end{aligned}$$

z	x_1	x_2	x_3	x_4	x_5	x'_4	x'_5	RHS
-1	2	3	1	0	0	M	M	0
0	1	4	2	-1	0	1	0	8
0	3	2	0	0	-1	0	1	6

z	x_1	x_2	x_3	x_4	x_5	x_4'	x_5'	RHS
-1	-4M +2	-6M+3	-2M+1	M	M	0	0	-14M
0	1	4	2	-1	0	1	0	8
0	3	2	0	0	-1	0	1	6

z	x_1	x_2	x_3	x_4	x_5	x_4'	x_5'	RHS
-1	-2.5M +1.25	0	M-0.5	-0.5M+0.75	M	1.5M-0.75	0	-2M-6
0	0.25	1	0.5	-0.25	0	0.25	0	2
0	2.5	0	-1	0.5	-1	-0.5	1	2

z	x_1	x_2	x_3	x_4	x_5	x_4'	x_5'	RHS
-1	0	0	0	0.5	0.5	M-0.5	M-0.5	-7
0	0	1	0.6	-0.3	0.1	0.3	-0.1	1.8
0	1	0	-0.4	0.2	-0.4	-0.2	0.4	0.8

Optimal Solution (x_1^*, x_2^*)=(0.8,1.8) and $z^* = 7$.

(2)

a.

z	x_1	x_2	x_3	x_4	x_5	x_6	x_4'	x_5'	x_6'	RHS
1	600	500	700	0	0	0	M	M	M	0
0	4	2	5	-1	0	0	1	0	0	24
0	5	6	2	0	-1	0	0	1	0	35
0	3	3	4	0	0	-1	0	0	1	30

z	x_1	x_2	x_3	x_4	x_5	x_6	x_4'	x_5'	x_6'	RHS
1	600-12M	500-11M	700-11M	M	M	M	0	0	0	-89M
0	4	2	5	-1	0	0	1	0	0	24
0	5	6	2	0	-1	0	0	1	0	35
0	3	3	4	0	0	-1	0	0	1	30

z	x_1	x_2	x_3	x_4	x_5	x_6	x_4'	x_5'	x_6'	RHS
1	0	-5M +200	4M -50	-2M +150	M	M	3M-150	0	0	-17M-3600
0	1	1/2	1 1/4	- 1/4	0	0	1/4	0	0	6
0	0	3 1/2	-4 1/4	1 1/4	-1	0	-1 1/4	1	0	5
0	0	1 1/2	1/4	3/4	0	-1	- 3/4	0	1	12

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	0	0	-29M/14 +2700/14	-3M/14 +1100/14	-3M/7 +400/7	M	17M/14- 1100/14	10M/7- 400/7	0	-69M/7- 27200/7
0	1	0	1 6/7	- 3/7	1/7	0	3/7	- 1/7	0	5 2/7
0	0	1	-1 3/14	5/14	- 2/7	0	- 5/14	2/7	0	1 3/7
0	0	0	2 1/14	3/14	3/7	-1	- 3/14	- 3/7	1	9 6/7

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	29M/26 - 2700/26	0	0	-126M/182 +22400/182	-49M/182 +7700/182	M	308M/182 -224/182	231M/182 - 7700/182	0	-721M/182 - 807100/182
0	7/13	0	1	- 3/13	1/13	0	3/13	- 1/13	0	2 11/13
0	17/26	1	0	1/13	- 5/26	0	- 1/13	5/26	0	4 23/26
0	-1 3/26	0	0	9/13	7/26	-1	- 9/13	- 7/26	1	3 25/26

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	94 4/9	0	0	0	-5 5/9	177 7/9	M	M + 50/9	M-1600/9	-5138 8/9
0	1/6	0	1	0	1/6	- 1/3	0	- 1/6	1/3	4 1/6
0	7/9	1	0	0	- 2/9	1/9	0	2/9	- 1/9	4 4/9
0	-1 11/18	0	0	1	7/18	-1 4/9	-1	- 7/18	1 4/9	5 13/18

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	71 3/7	0	0	14 2/7	0	157 1/7	M-100/7	M	M-1100/7	-5057 1/7
0	6/7	0	1	- 3/7	0	2/7	3/7	0	- 2/7	1 5/7
0	- 1/7	1	0	4/7	0	- 5/7	- 4/7	0	5/7	7 5/7
0	-4 1/7	0	0	2 4/7	1	-3 5/7	-2 4/7	-1	3 5/7	14 5/7

Optimal Solution (x_1^*, x_2^*, x_3^*)=(0, 7 5/7,1 5/7) and z^*=5057 1/7 .

b.

Phase 1

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	0	0	0	0	0	0	1	1	1	0
0	4	2	5	-1	0	0	1	0	0	24
0	5	6	2	0	-1	0	0	1	0	35
0	3	3	4	0	0	-1	0	0	1	30

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	-12	-11	-11	1	1	1	0	0	0	-89
0	4	2	5	-1	0	0	1	0	0	24
0	5	6	2	0	-1	0	0	1	0	35
0	3	3	4	0	0	-1	0	0	1	30

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	0	-5	4	-2	1	1	3	0	0	-17
0	1	1/2	1 1/4	- 1/4	0	0	1/4	0	0	6
0	0	3 1/2	-4 1/4	1 1/4	-1	0	-1 1/4	1	0	5
0	0	1 1/2	1/4	3/4	0	-1	- 3/4	0	1	12

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	0	0	-2 1/14	- 3/14	- 3/7	1	1 3/14	1 3/7	0	-9 6/7
0	1	0	1 6/7	- 3/7	1/7	0	3/7	- 1/7	0	5 2/7
0	0	1	-1 3/14	5/14	- 2/7	0	- 5/14	2/7	0	1 3/7
0	0	0	2 1/14	3/14	3/7	-1	- 3/14	- 3/7	1	9 6/7

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	1 3/26	0	0	- 9/13	- 7/26	1	1 9/13	1 7/26	0	-3 25/26
0	7/13	0	1	- 3/13	1/13	0	3/13	- 1/13	0	2 11/13
0	17/26	1	0	1/13	- 5/26	0	- 1/13	5/26	0	4 23/26
0	-1 3/26	0	0	9/13	7/26	-1	- 9/13	- 7/26	1	3 25/26

z	x_1	x_2	x_3	x_4	x_5	x_6	x'_4	x'_5	x'_6	RHS
1	0	0	0	0	0	0	1	1	1	0
0	1/6	0	1	0	1/6	- 1/3	0	- 1/6	1/3	4 1/6
0	7/9	1	0	0	- 2/9	1/9	0	2/9	- 1/9	4 4/9
0	-1 11/18	0	0	1	7/18	-1 4/9	-1	- 7/18	1 4/9	5 13/18

Phase 2

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	600	500	700	0	0	0	0
0	1/6	0	1	0	1/6	- 1/3	4 1/6
0	7/9	1	0	0	- 2/9	1/9	4 4/9
0	-1 11/18	0	0	1	7/18	-1 4/9	5 13/18

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	94 4/9	0	0	0	-5 5/9	177 7/9	-5138 8/9
0	1/6	0	1	0	1/6	- 1/3	4 1/6
0	7/9	1	0	0	- 2/9	1/9	4 4/9
0	-1 11/18	0	0	1	7/18	-1 4/9	5 13/18

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	71 3/7	0	0	14 2/7	0	157 1/7	-5057 1/7
0	6/7	0	1	- 3/7	0	2/7	1 5/7
0	- 1/7	1	0	4/7	0	- 5/7	7 5/7
0	-4 1/7	0	0	2 4/7	1	-3 5/7	14 5/7

(3) Phase 1

z	x_1	x_2	x_3	x_4	x_5	x_7	x_5'	x_6'	RHS
1	0	0	0	0	0	0	1	1	0
0	1	-2	1	0	-1	0	1	0	20
0	2	4	1	0	0	0	0	1	50
0	1	0	0	1	0	1	0	0	10

z	x_1	x_2	x_3	x_4	x_5	x_7	x_5'	x_6'	RHS
1	-3	-2	-2	0	1	0	0	0	-70
0	1	-2	1	0	-1	0	1	0	20
0	2	4	1	0	0	0	0	1	50
0	1	0	0	1	0	1	0	0	10

z	x_1	x_2	x_3	x_4	x_5	x_7	x_5'	x_6'	RHS
1	0	-2	-2	3	1	3	0	0	-40
0	0	-2	1	-1	-1	-1	1	0	10
0	0	4	1	-2	0	-2	0	1	30
0	1	0	0	1	0	1	0	0	10

z	x_1	x_2	x_3	x_4	x_5	x_7	x_5'	x_6'	RHS
1	0	0	-1 1/2	2	1	2	0	1/2	-25
0	0	0	1 1/2	-2	-1	-2	1	1/2	25
0	0	1	1/4	- 1/2	0	- 1/2	0	1/4	7 1/2
0	1	0	0	1	0	1	0	0	10

z	x_1	x_2	x_3	x_4	x_5	x_7	x_5'	x_6'	RHS
1	0	0	0	0	0	0	1	1	0
0	0	0	1	-1 1/3	- 2/3	-1 1/3	2/3	1/3	16 2/3
0	0	1	0	- 1/6	1/6	- 1/6	- 1/6	1/6	3 1/3
0	1	0	0	1	0	1	0	0	10

Phase 2

z	x_1	x_2	x_3	x_4	x_5	x_7	RHS
1	-2	-5	-3	-1	0	0	0
0	0	0	1	-1 1/3	- 2/3	-1 1/3	16 2/3
0	0	1	0	- 1/6	1/6	- 1/6	3 1/3
0	1	0	0	1	0	1	10

z	x_1	x_2	x_3	x_4	x_5	x_7	RHS
1	0	0	0	-3 5/6	-1 1/6	-2 5/6	86 2/3
0	0	0	1	-1 1/3	- 2/3	-1 1/3	16 2/3
0	0	1	0	- 1/6	1/6	- 1/6	3 1/3
0	1	0	0	1	0	1	10

z	x_1	x_2	x_3	x_4	x_5	x_7	RHS
1	3 5/6	0	0	0	-1 1/6	1	125
0	1 1/3	0	1	0	- 2/3	0	30
0	1/6	1	0	0	1/6	0	5
0	1	0	0	1	0	1	10

z	x_1	x_2	x_3	x_4	x_5	x_7	RHS
1	5	7	0	0	0	1	160
0	2	4	1	0	0	0	50
0	1	6	0	0	1	0	30
0	1	0	0	1	0	1	10

Optimal Solution $(x_1^*, x_2^*, x_3^*, x_4^*)$=(0, 0, 50,10) and $z^* = 160$.

(4)

a. False. If the problem is unbounded, then the best corner point is not optimal solution.

b. $x_2 = -x_2' \quad where \quad x_2' \geq 0$

$x_3 = x_3^+ - x_3^- \quad where \quad x_3^+, x_3^- \geq 0$

$x_1 + x_2' + 2(x_3^+ - x_3^-) - x_4 = 1$ and $x_4 \geq 0$

c. False. For a problem with alternative optimal solutions or in case of degeneracy, the objective function value might stay the same.
d. If Big-M method leads to an optimal solution with an objective that is a function of M or if the first phase of two-phase method leads to a non-zero answer, the problem is infeasible.
e. False. Depending on objective function's direction it may not be increased indefinitely. e.g., $\max -x_1$ $s.t.$ $x_1 \geq 0$ has an unbounded feasible region but a bounded objective function.
(5)
Phase 1

z	x_1	x_2	x_3	x_4	x_5	x_5'	RHS
-1	0	0	0	0	0	1	0
0	1	2	-1	-1	0	1	5
0	-3	-1	1	0	1	0	4

z	x_1	x_2	x_3	x_4	x_5	x_5'	RHS
-1	-1	-2	1	1	0	0	-5
0	1	2	-1	-1	0	1	5
0	-3	-1	1	0	1	0	4

z	x_1	x_2	x_3	x_4	x_5	x_5'	RHS
-1	0	0	0	0	0	1	0
0	0.5	1	-0.5	-0.5	0	0.5	2.5
0	-2.5	0	0.5	-0.5	1	0.5	6.5

Phase 2

z	x_1	x_2	x_3	x_4	x_5	RHS
-1	3	-3	1	0	0	0
0	0.5	1	-0.5	-0.5	0	2.5
0	-2.5	0	0.5	-0.5	1	6.5

z	x_1	x_2	x_3	x_4	x_5	RHS
-1	4.5	0	-0.5	-1.5	0	7.5
0	0.5	1	-0.5	-0.5	0	2.5
0	-2.5	0	0.5	-0.5	1	6.5

The problem is unbounded.

(6) Yes, it is possible if the objective function is parallel to a constraint and the region is unbounded in an opposite direction. e.g., max $-x_1$ $s.t.$ $x_1 \geq 0, x_2 \geq 0$ has an unbounded feasible region but has alternative optima.

(7)

z	x_1	x_2	x_3	x_4'	RHS
1	-2	-3	0	M	0
0	1	2	1	0	4
0	1	1	0	1	3

z	x_1	x_2	x_3	x_4'	RHS
1	-M-2	-M-3	0	0	-3M
0	1	2	1	0	4
0	1	1	0	1	3

z	x_1	x_2	x_3	x_4'	RHS
1	-M/2-1/2	0	M/2+3/2	0	-M+6
0	0.5	1	0.5	0	2
0	0.5	0	-0.5	1	1

z	x_1	x_2	x_3	x_4'	RHS
1	0	0	1	M+1	7
0	0	1	1	-1	1
0	1	0	-1	2	2

Optimal Solution (x_1^*, x_2^*)=(2,1) and $z^* = 7$.

6.3 Chapter 3 Exercises

(1)

$$c_B B^{-1} N - c_N = \begin{bmatrix} c_1 \; 2 \; c_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ -1/2 & 1 & 3/2 \\ 3/2 & -2 & -5/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \; 0 \; 0 \end{bmatrix} = \begin{bmatrix} f_{04} \; 3 \; 7/2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 - 1 + 3c_3/2 \; 2 - c_1 - 2c_3 \; 3 - 2c_1 - 5c_3/2 \end{bmatrix} = \begin{bmatrix} f_{04} \; 3 \; 7/2 \end{bmatrix}$$

From the equation, $c_1 = 1, c_3 = -1$.

$x_B : x_1, x_2, x_3$ is given and we know from the question that x_4, x_2, x_3 will be basic variables at the next iteration. It means that x_1 will leave and x_4 will enter. Then the pivot is in the first row of x_4 column.

$$B^{-1} = EB^{-1}_{old} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ -1/2 & 1 & 3/2 \\ 3/2 & -2 & -5/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 5 \end{bmatrix}$$

$$c_B B^{-1} b = \begin{bmatrix} 0 \; 2 \; -1 \end{bmatrix} \begin{bmatrix} 10 \\ 15 \\ 5 \end{bmatrix} = 25$$

Optimal solution is $x_1^* = 0, x_2^* = 15, x_3^* = 5, z^* = 25$

(2)

a.

$x_B = x_1, x_2, x_6, x_7$

$$B = \begin{bmatrix} 8 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B^{=1} = \begin{bmatrix} 3/16 & -1/8 & 0 & 0 \\ -1/4 & 1/2 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} 3/16 & -1/8 & 0 & 0 \\ -1/4 & 1/2 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \\ 50 \\ 20 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \\ 0 \\ 20 \end{bmatrix}$$

It is a basic feasible solution.

b.

$$c_B B^{-1} N - c_N = \begin{bmatrix} 20 & 6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3/16 & -1/8 & 0 & 0 \\ -1/4 & 1/2 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & 9/4 & 1/2 \end{bmatrix}$$

$$c_B B^{-1} b = \begin{bmatrix} 20 & 6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 25 \\ 0 \\ 0 \\ 20 \end{bmatrix} = 500$$

Optimal solution is $x_1^* = 25, x_2^* = 0, x_3^* = 0, z^* = 500$

(3) a.

$x_B = x_1, x_2, x_3$

$$B = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 4 & 2 \\ 3 & 1 & -1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} -6/17 & 1/17 & 8/17 \\ 5/17 & 2/17 & -1/17 \\ -13/17 & 5/17 & 6/17 \end{bmatrix} \quad B^{-1}b = \begin{bmatrix} -6/17 & 1/17 & 8/17 \\ 5/17 & 2/17 & -1/17 \\ -13/17 & 5/17 & 6/17 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 25/17 \\ 16/17 \\ 23/17 \end{bmatrix}$$

it is feasible.

(b)

$$c_BB^{-1}N - c_N = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} -6/17 & 1/17 & 8/17 \\ 5/17 & 2/17 & -1/17 \\ -13/17 & 5/17 & 6/17 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \geq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Then,

$$\begin{bmatrix} -6c_1 + 5c_2 - 13c_3 & -c_1 - 2c_2 - 5c_3 & 8c_1 - 5c_2 + 6c_3 \end{bmatrix} \geq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

From the inequality above we get, $c_1 \geq 0, c_2 \geq 0, c_3 \leq 0 \quad and \quad c_1 \geq -3c_3$. As an example, when $c_1 = 3, c_2 = 1, c_3 = -1$ this iteration gives optimal solution.

(c)

$$c_BB^{-1}N - c_N = \begin{bmatrix} 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} -6/17 & -1/17 & 8/17 \\ 5/17 & -2/17 & -1/17 \\ -13/17 & -5/17 & 6/17 \end{bmatrix} = \begin{bmatrix} 7/6 & -1/2 & -1/4 \end{bmatrix}$$

It is not optimal and x_5 must enter but x_5 column of the matrix $B^{-1}N$ has negative numbers. So, optimal solution is unbounded.

(4)

$$B^{-1} = \begin{bmatrix} 0.30 & -0.10 \\ -0.20 & 0.40 \end{bmatrix} \qquad B^{-1} \begin{bmatrix} 1 & 4 & 2 & -1 & 0 \\ 3 & 2 & 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0.40 & -0.30 & 0.10 \\ 1 & 0 & 0.40 & 0.20 & -0.40 \end{bmatrix}$$

x_1 and x_2 are basic variables.

$$c_BB^{-1}N - c_N = \begin{bmatrix} -3 & -2 \end{bmatrix} \begin{bmatrix} 0.40 & -0.30 & 0.10 \\ 0.40 & 0.20 & -0.40 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} 0.30 & -0.10 \\ -0.20 & 0.40 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 0.8 \end{bmatrix} \qquad c_BB^{-1}b = \begin{bmatrix} -3 & -2 \end{bmatrix} \begin{bmatrix} 1.8 \\ 0.8 \end{bmatrix} = -7$$

z	x_1	x_2	x_3	x_4	$\bar{x_5}$	x_6	$\bar{x_7}$	RHS
1	0	0	0	0.5	M - 0.5	0.5	M-0.5	-7
0	0	1	0.4	-0.3	0.3	0.1	-0.1	1.8
0	1	0	0.4	0.2	-0.2	-0.4	0.4	0.8

(5)

$x_B = x_1, x_2, x_6$

$x_N = x_3, x_4, x_5$

$$B = \begin{bmatrix} 1\ 4\ 0 \\ 3\ 2\ 0 \\ 1\ 1\ 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} -1/5 & 2/5 & 0 \\ 3/10 & -1/10 & 0 \\ -1/10 & -3/10 & 1 \end{bmatrix} \quad B^{-1}b = \begin{bmatrix} -1/5 & 2/5 & 0 \\ 3/10 & -1/10 & 0 \\ -1/10 & -3/10 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 9/5 \\ 12/5 \end{bmatrix} Feasible$$

$$c_B B^{-1} N - c_N = \begin{bmatrix} -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} -1/5 & 2/5 & 0 \\ 3/10 & -1/10 & 0 \\ -1/10 & -3/10 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1/2 & 1/2 \end{bmatrix} x_3 \quad enters$$

$$B^{-1}N = \begin{bmatrix} -2/5 & 1/5 & -2/5 \\ 3/5 & -3/10 & 1/10 \\ 4/5 & 1/10 & 3/10 \end{bmatrix} \min\{\frac{9/5}{3/5}, \frac{12/5}{4/5}\} = 3$$

Because both give 3, we can choose any of them. Let's assume that x_2 leaves.

<u>Next Iteration</u>

$x_B = x_1, x_3, x_6$

$x_N = x_2, x_4, x_5$

$$B^{-1} = EB^{-1}_{old} = \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 5/3 & 0 \\ 0 & -4/3 & 1 \end{bmatrix} \begin{bmatrix} -1/5 & 2/5 & 0 \\ 3/10 & -1/10 & 0 \\ -1/10 & -3/10 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 0 \\ 1/2 & -1/6 & 0 \\ -1/2 & -1/6 & 1 \end{bmatrix}$$

$$c_B B^{-1} N - c_N = \begin{bmatrix} -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 0 \\ 1/2 & -1/6 & 0 \\ -1/2 & -1/6 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 & 0 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 10/3 & -1/2 & 5/6 \end{bmatrix} x_4 \quad enters$$

$$B^{-1}b = \begin{bmatrix} 0 & 1/3 & 0 \\ 1/2 & -1/6 & 0 \\ -1/2 & -1/6 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$B^{-1}N = \begin{bmatrix} 2/3 & 0 & -1/3 \\ 5/3 & -1/2 & 1/6 \\ -4/3 & 1/2 & 1/6 \end{bmatrix} \min\{\frac{0}{1/2}\} = 0 \quad x_6 leaves.$$

<u>Next Iteration</u>

$x_B = x_1, x_3, x_4$

$x_N = x_2, x_6, x_5$

$$B^{-1} = EB_{old}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 0 \\ 1/2 & -1/6 & 0 \\ -1/2 & -1/6 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 0 \\ 0 & -1/3 & 1 \\ -1 & -1/3 & 2 \end{bmatrix}$$

$$c_B B^{-1} N - c_N = \begin{bmatrix} -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 0 \\ 0 & -1/3 & 1 \\ -1 & -1/3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} Optimal$$

$$B^{-1}b = \begin{bmatrix} 0 & 1/3 & 0 \\ 0 & -1/3 & 1 \\ -1 & -1/3 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \quad c_B B^{-1} b = \begin{bmatrix} -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = -1$$

Optimal solution is $x_1^* = 2, x_2^* = 0, x_3^* = 3, z^* = 1$

(6)

$x_B = x_1, x_2, x_6, x_7$

$x_N = x_3, x_4, x_5$

$$B = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} -1/5 & 2/5 & 0 & 0 \\ 3/10 & -1/10 & 0 & 0 \\ -1/10 & -3/10 & 1 & 0 \\ 1/5 & -2/5 & 0 & 1 \end{bmatrix} \quad B^{-1}b = \begin{bmatrix} -1/5 & 2/5 & 0 & 0 \\ 3/10 & -1/10 & 0 & 0 \\ -1/10 & -3/10 & 1 & 0 \\ 1/5 & -2/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 9/5 \\ 12/5 \\ 16/5 \end{bmatrix} Feasible$$

$$c_B B^{-1} N - c_N = \begin{bmatrix} -2 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/5 & 2/5 & 0 & 0 \\ 3/10 & -1/10 & 0 & 0 \\ -1/10 & -3/10 & 1 & 0 \\ 1/5 & -2/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1/2 & 1/2 \end{bmatrix} x_3 \quad enters$$

$$B^{-1}N = \begin{bmatrix} -2/5 & 1/5 & -2/5 \\ 3/5 & -3/10 & 1/10 \\ 4/5 & 1/10 & 3/10 \\ 2/5 & -1/5 & 2/5 \end{bmatrix} \min\{\frac{9/5}{3/5}, \frac{12/5}{4/5}, \frac{16/5}{2/5}\} = 3$$

Because both give 3, we can choose any of them. Let's assume that x_2 leaves.

<u>Next Iteration</u>

$x_B = x_1, x_3, x_6, x_7$

$x_N = x_2, x_4, x_5$

$$B^{-1} = EB^{-1}_{old} = \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 5/3 & 0 & 0 \\ 0 & -4/3 & 1 & 0 \\ 0 & -2/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/5 & 2/5 & 0 & 0 \\ 3/10 & -1/10 & 0 & 0 \\ -1/10 & -3/10 & 1 & 0 \\ 1/5 & -2/5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1/2 & -1/6 & 0 & 0 \\ -1/2 & -1/6 & 1 & 0 \\ 0 & -1/3 & 0 & 1 \end{bmatrix}$$

$$c_B B^{-1} N - c_N = \begin{bmatrix} -2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1/2 & -1/6 & 0 & 0 \\ -1/2 & -1/6 & 1 & 0 \\ 0 & -1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 & 0 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 10/3 & -1/2 & 5/6 \end{bmatrix} x_4 \quad enters$$

$$B^{-1}b = \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1/2 & -1/6 & 0 & 0 \\ -1/2 & -1/6 & 1 & 0 \\ 0 & -1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

$$B^{-1}N = \begin{bmatrix} 2/3 & 0 & -1/3 \\ 5/3 & -1/2 & 1/6 \\ -4/3 & 1/2 & 1/6 \\ -2/3 & 0 & 1/3 \end{bmatrix} \min\{\frac{0}{1/2}\} = 0 \quad x_6 leaves.$$

<u>Next Iteration</u>

$x_B = x_1, x_3, x_4, x_7$

$x_N = x_2, x_6, x_5$

$$B^{-1} = EB_{old}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1/2 & -1/6 & 0 & 0 \\ -1/2 & -1/6 & 1 & 0 \\ 0 & -1/3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 0 & -1/3 & 1 & 0 \\ -1 & -1/3 & 2 & 0 \\ 0 & -1/3 & 0 & 1 \end{bmatrix}$$

$$c_B B^{-1} N - c_N = \begin{bmatrix} -2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 0 & -1/3 & 1 & 0 \\ -1 & -1/3 & 2 & 0 \\ 0 & -1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} Optimal$$

$$B^{-1}b = \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 0 & -1/3 & 1 & 0 \\ -1 & -1/3 & 2 & 0 \\ 0 & -1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \end{bmatrix} \quad c_B B^{-1} b = \begin{bmatrix} -2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \end{bmatrix} = -1$$

Optimal solution is $x_1^* = 2, x_2^* = 0, x_3^* = 3, z^* = 1$

6.4 Chapter 4 Exercises

(1)

$$
\begin{aligned}
\max\ & -3y_1 + 2y_2 \\
\text{s.t.}\ & -2y_1 + y_2 \leq 2 \\
& 5y_1 + 6y_2 \leq 15 \\
& -4y_1 + 3y_2 \leq 5 \\
& 3y_1 + y_2 \leq 6 \\
& y_1 \leq 0, y_2 \geq 0
\end{aligned}
$$

(2)

a.

$$
\begin{aligned}
\min\ & 10y_1 + 5y_2 \\
\text{s.t.}\ & y_1 + 3y_2 \leq 4 \\
& y_1 + y_2 \geq 1 \\
& y_1 \geq 0, y_2 \leq 0
\end{aligned}
$$

b.

Primal						Dual					
x_1	x_2	x_3	x_4	z	Feasible?	y_1	y_2	y_3	y_4	z	Feasible?
0	0	10	-5	0	Infeasible	0	0	4	-1	0	Infeasible
0	10	0	5	10	Feasible	1	0	3	0	10	Feasible
0	5	5	0	5	Feasible	0	1	1	0	5	Infeasible
10	0	0	25	40	Infeasible	4	0	0	3	40	Feasible
5/3	0	25/3	0	20/3	Infeasible	0	4/3	0	1/3	20/3	Infeasible
-5/2	25/2	0	0	5/2	Feasible	-1/2	3/2	0	0	5/2	Infeasible

Optimal solution is $x_1^* = 0, x_2^* = 10, z^* = 10$

(3)

a. The procedure is straightforward and omitted here for brevity.

Final tableau:

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
-1	2/3	2/3	0	1/3	0	2/3	-10/3
0	1/3	4/3	0	5/3	1	1/3	19/3
0	2/3	- 1/3	1	- 2/3	0	- 1/3	5/3

b.

$B^{-1} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix}$. It is easy to see that when b_1 changes from 8 to 1, the basis is not feasible as $B^{-1}\begin{bmatrix} 1 \\ 5 \end{bmatrix} \ngeq 0$. Next, you need to apply dual simplex!

(4)

$$\begin{aligned} &\min\ 5y_1 + 4y_2 + 6y_3 \\ &\text{s.t.}\ \ -2y_1 + 2y_2 \leq -2 \\ &\qquad y_1 - 2y_3 \geq 3 \\ &\qquad 3y_1 + y_2 + y_3 \geq 5 \\ &\qquad y_1 + y_3 = 0 \\ &\qquad y_1 \leq 0, y_2 urs, y_3 \geq 0 \end{aligned}$$

Primal								Dual							
x_1	x_2	x_3	x_4	x_5	x_6	z	Feasible?	y_1	y_2	y_3	y_4	y_5	y_6	z	Feasible?
0	0	4	-7	0	9	20	Feasible	0	5	0	-12	-3	0	20	Infeasible
0	0	4	2	9	0	20	Feasible	0	5	0	-12	-3	0	20	Infeasible
0	-3	4	-4	0	0	11	Infeasible	1	3	-1	-6	0	0	11	Infeasible
2	0	0	9	0	-3	-4	Infeasible	0	-1	0	0	-3	-6	-4	Infeasible
2	0	0	6	-3	0	-4	Infeasible	0	-1	0	0	-3	-6	-4	Infeasible
3/2	0	1	5	0	0	2	Infeasible	2	1	-2	0	3	0	2	Infeasible
2	1	0	8	0	0	-1	Infeasible	-1	-2	1	0	0	-9	-7	Infeasible

Optimal solution goes infinity.

(5)

a.

$$\begin{aligned} &\min\ -x_1 + c_2x_2 - x_3 \\ &\text{s.t.}\ \ -x_1 - x_2 + x_3 \leq 1 \end{aligned}$$

$$
\begin{aligned}
&2x_1 - x_2 + x_3 \le 2 \\
&x_1 \ge -2 \\
&x_1 \le 1 \\
&x_3 \ge -2 \\
&x_3 \le -1 \\
&x_1, x_2, x_3 urs
\end{aligned}
$$

$$
\begin{aligned}
\max\ & y_1 + 2y_2 - 2y_3 + y_4 - 2y_5 - y_6 \\
\text{s.t.}\ & -y_1 + 2y_2 + y_3 + y_4 = -1 \\
& -y_1 - y_2 = c_2 \\
& y_1 + y_2 + y_5 + y_6 = -1 \\
& y_1, y_2, y_4, y_6 \le 0, y_3, y_5 \ge 0
\end{aligned}
$$

b.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
-1	2	-1	3	7	1	2	1	0

The solution is feasible but it is not a feasible corner point. For it to be a feasible corner points there must be 3 non-basic variables but there is only one non-basic variable. That's why this solution cannot be a optimal solution and doesn't depend on value of c_2.

c.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
1/3	-10/3	-2	0	0	7/3	2/3	0	1

This solution is a feasible and a corner point. To check whether it is optimal or not we should calculate its corresponding dual variables.

y_1	y_2	y_3	y_4	y_5	y_6
$\frac{1-2c_2}{3}$	$\frac{-1-c_2}{3}$	0	0	-1+c_2	0

This solution can be optimal if necessary conditions are provided for c_2.

$y_1 = \frac{1-2c_2}{3} \le 0$, $c_2 \ge 1/2$

$y_2 = \frac{-1-c_2}{3} \le 0$, $c_2 \ge -1$

$y_5 = -1 + c_2 \ge 0$, $c_2 \ge 1$

While $c_2 \ge 1$, this solution is optimal solution.

d.

Primal									Dual					
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6
1/3	- 7/3	-1	0	0	7/3	2/3	1	0	-1/3	-2/3	0	0	0	0

(6)

If P is infeasible, then D is infeasible or unbounded. If there exists a feasible solution for D, then D cannot be infeasible. Thus, we can conclude that D is unbounded.

6.5 Chapter 5 Exercises

(1)

a.

$$B^{-1}b = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 90 \end{bmatrix} = \begin{bmatrix} 30 \\ -30 \end{bmatrix}, c_B B^{-1}b = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 30 \\ -30 \end{bmatrix} = 150$$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	2	5	0	150
0	-1	1	3	1	0	30
0	16	0	-2	-4	1	-30

z	x_1	x_2	x_3	x_4	x_5	RHS
1	16	0	0	1	1	120
0	23	1	0	-5	3	-15
0	-8	0	1	2	-1/2	15

z	x_1	x_2	x_3	x_4	x_5	RHS
1	57/5	1/5	0	0	8/5	117
0	-23/5	-1/5	0	1	-3/5	3
0	6/5	-2/5	1	0	1/5	9

Optimal solution is $x_1^* = 0, x_2^* = 0, x_3^* = 9, z^* = 117$

b.

$$B^{-1}b = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 70 \end{bmatrix} = \begin{bmatrix} 20 \\ -10 \end{bmatrix}, c_B B^{-1} b = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 20 \\ -10 \end{bmatrix} = 100$$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	2	5	0	100
0	-1	1	3	1	0	20
0	16	0	-2	-4	1	-10

z	x_1	x_2	x_3	x_4	x_5	RHS
1	16	0	0	1	1	90
0	23	1	0	-5	3	5
0	-8	0	1	2	-1	5

Optimal solution is $x_1^* = 0, x_2^* = 5, x_3^* = 5, z^* = 90$

c.

$$B^{-1}b = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 100 \end{bmatrix} = \begin{bmatrix} 10 \\ 60 \end{bmatrix}, \text{ feasible}, c_B B^{-1} b = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 60 \end{bmatrix} = 50$$

d.

$$c_B B^{-1} A_3 - c_3 = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} - 8 = 7, \text{ optimal}$$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	7	5	0	100
0	-1	1	3	1	0	20
0	16	0	-2	-4	1	10

e.

$$c_B B^{-1} A_1 - c_1 = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} - (-2) = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} + 2 = 2, \text{ optimal}$$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	2	0	2	5	0	100
0	0	1	3	1	0	20
0	5	0	-2	-4	1	10

f.

$$c_B B^{-1} A_2 - c_2 = \begin{bmatrix} 5 \; 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} - 6 = \begin{bmatrix} 5 \; 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} - 6 = 4$$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	4	2	5	0	100
0	-1	2	3	1	0	20
0	16	-3	-2	-4	1	10

z	x_1	x_2	x_3	x_4	x_5	RHS
1	2	0	-4	3	0	60
0	-1/2	1	3/2	1/2	0	10
0	29/2	0	5/2	-5/2	1	40

z	x_1	x_2	x_3	x_4	x_5	RHS
1	2/3	8/3	0	13/3	0	260/3
0	- 1/3	2/3	1	1/3	0	20/3
0	46/3	-5/3	0	-10/3	1	70/3

Optimal solution is $x_1^* = 0, x_2^* = 0, x_3^* = 20/3, z^* = 260/3$

g.

$$c_B B^{-1} A_6 - c_6 = \begin{bmatrix} 5 \; 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} - 10 = \begin{bmatrix} 5 \; 0 \end{bmatrix} \begin{bmatrix} 3 \\ -7 \end{bmatrix} - 10 = 5, \text{ optimal.}$$

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	2	5	0	5	100
0	-1	1	3	1	0	3	20
0	16	0	-2	-4	1	-7	10

h.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	2	5	0	0	100
0	-1	1	3	1	0	0	20
0	16	0	-2	-4	1	0	10
0	2	3	5	0	0	1	50

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	2	5	0	0	100
0	-1	1	3	1	0	0	20
0	16	0	-2	-4	1	0	10
0	5	0	-4	-3	0	1	-10

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	5/2	0	0	7/2	0	1/2	95
0	1/4	1	0	-5/4	0	3/4	25/2
0	27/2	0	0	-5/2	1	-1/2	15
0	-5/4	0	1	3/4	0	-1/4	5/2

Optimal solution is $x_1^* = 0, x_2^* = 25/2, x_3^* = 5/2, z^* = 95$

i.

$$B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$$

$$B^{-1}N = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 1 \\ 10 & 10 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 1 \\ 15 & -5 & -5 \end{bmatrix}$$

$$c_B B^{-1} N - c_N = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 & 1 \\ 15 & -5 & -5 \end{bmatrix} - \begin{bmatrix} -5 & 13 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 5 \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 100 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}, c_B B^{-1} b = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 20 \\ 0 \end{bmatrix} = 100$$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	2	5	0	100
0	-1	1	3	1	0	20
0	15	0	-5	-5	1	0

Optimal solution is $x_1^* = 0, x_2^* = 20, x_3^* = 0, z^* = 100$

(2)

a.

$$B^{-1}b = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 70 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 30 \\ 15 \\ -5 \end{bmatrix}, c_B B^{-1} b = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 30 \\ 15 \\ -5 \end{bmatrix} = 35$$

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	3/2	0	3/2	1/2	35
0	0	0	1	1	-1	-2	30
0	1	0	1/2	0	1/2	1/2	15
0	0	1	-3/2	0	-1/2	1/2	-5

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	1	0	0	1	1	30
0	0	2/3	0	1	-4/3	-5/3	80/3
0	1	1/3	0	0	1/3	2/3	40/3
0	0	-2/3	1	0	1/3	-1/3	10/3

Optimal solution is $x_1^* = 40/3, x_2^* = 0, x_3^* = 10/3, z^* = 30$

b.

$$B^{-1}A_1 = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

x_4, x_1, x_2 cannot construct a basis. Their constraints are linearly dependent. We have to solve the problem from the scratch.

c.

$$B^{-1}A_3 = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1/2 \\ -3/2 \end{bmatrix}$$

$$c_B B^{-1} A_3 - c_3 = \begin{bmatrix} 0 \; 2 \; -1 \end{bmatrix} \begin{bmatrix} 6 \\ -1/2 \\ -3/2 \end{bmatrix} - 2 = -3/2$$

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	-3/2	0	3/2	1/2	25
0	0	0	6	1	-1	-2	10
0	1	0	-1/2	0	1/2	1/2	15
0	0	1	-3/2	0	-1/2	1/2	5

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	0	1/4	5/4	0	55/2
0	0	0	1	1/6	-1/6	-1/3	5/3
0	1	0	0	1/12	5/12	1/3	95/6
0	0	1	0	1/4	-3/4	0	15/2

Optimal solution is $x_1^* = 95/6, x_2^* = 15/2, x_3^* = 5/3, z^* = 55/2$

d.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	-3	2	-3	0	0	0	0
0	0	0	1	1	-1	-2	10
0	1	0	1/2	0	1/2	1/2	15
0	0	1	-3/2	0	-1/2	1/2	5

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	3/2	0	5/2	1/2	35
0	0	0	1	1	-1	-2	10
0	1	0	1/2	0	1/2	1/2	15
0	0	1	-3/2	0	-1/2	1/2	5

e.

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
1	0	0	3/2	0	3/2	1/2	0	25
0	0	0	1	1	-1	-2	0	10
0	1	0	1/2	0	1/2	1/2	0	15
0	0	1	-3/2	0	-1/2	1/2	0	5
0	3	-2	1	0	0	0	1	30

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
1	0	0	3/2	0	3/2	1/2	0	25
0	0	0	1	1	-1	-2	0	10
0	1	0	1/2	0	1/2	1/2	0	15
0	0	1	-3/2	0	-1/2	1/2	0	5
0	0	0	-7/2	0	-5/2	-1/2	1	-5

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
1	0	0	0	0	3/7	2/7	3/7	160/7
0	0	0	0	1	-12/7	-15/7	2/7	40/7
0	1	0	0	0	1/7	3/7	1/7	100/7
0	0	1	0	0	4/7	5/7	-3/7	50/7
0	0	0	1	0	5/7	1/7	-2/7	10/7

Optimal solution is $x_1^* = 100/7, x_2^* = 50/7, x_3^* = 10/7, z^* = 160/7$

f.

$$c_B B^{-1} A_8 - c_8 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} - (-1) = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ -1.5 \end{bmatrix} + 1 = 1.5, \text{ optimal.}$$

(3)

a.

$$B^{-1}b = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} -10 \\ 30 \end{bmatrix}, c_B B^{-1} b = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} -10 \\ 30 \end{bmatrix} = 60$$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	1	1	0	2	60
0	0	-1	5	1	-1	-10
0	1	4	-1	0	1	30

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	6	1	1	50
0	0	1	-5	-1	1	10
0	1	0	19	4	-3	-10

z	x_1	x_2	x_3	x_4	x_5	RHS
1	1/3	0	37/3	7/3	0	140/3
0	1/3	1	4/3	1/3	0	20/3
0	-1/3	0	-19/3	-4/3	1	10/3

Optimal solution is $x_1^* = 0, x_2^* = 20/3, x_3^* = 0, z^* = 140/3$

b.

$$c_B B^{-1} A_3 - c_3 = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} - (-2) = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} + 2 = -2, \text{ not optimal (suboptimal).}$$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	1	-2	0	2	20
0	0	-1	5	1	-1	20
0	1	4	-2	0	1	10

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	3/5	0	2/5	8/5	28
0	0	-1/5	1	1/5	-1/5	4
0	1	18/5	-0	2/5	3/5	18

Optimal solution is $x_1^* = 18, x_2^* = 0, x_3^* = 0, z^* = 28$

c.

$$c_B B^{-1} A_1 - c_1 = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 4 = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4 = 0$$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	1	1	0	2	20
0	1	-1	5	1	-1	20
0	2	4	-1	0	1	10

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	1	1	0	2	20
0	0	-3	11/2	1	-3/2	15
0	1	2	-1/2	0	1/2	5

Optimal solution is $x_1^* = 5, x_2^* = 0, x_3^* = 0, z^* = 20$

e.

z	x_1	x_2	x_3	x_4	x_5	RHS
1	-1	-5	2	0	0	0
0	0	-1	5	1	-1	20
0	1	4	-1	0	1	10

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	-1	1	0	1	10
0	0	-1	5	1	-1	20
0	1	4	-1	0	1	10

z	x_1	x_2	x_3	x_4	x_5	RHS
1	1/4	0	3/4	0	5/4	25/2
0	1/4	0	19/4	1	-3/4	45/2
0	1/4	1	-1/4	0	1/4	5/2

Optimal solution is $x_1^* = 0, x_2^* = 5/2, x_3^* = 0, z^* = 25/2$

f.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	1	1	0	2	0	20
0	0	-1	5	1	-1	0	20
0	1	4	-1	0	1	0	10
0	3	2	3	0	0	1	25

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	1	1	0	2	0	20
0	0	-1	5	1	-1	0	20
0	1	4	-1	0	1	0	10
0	0	-10	6	0	-3	1	-5

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	8/5	0	17/10	1/10	39/2
0	0	0	22/5	1	-7/10	-1/10	41/2
0	1	0	7/5	0	-1/5	2/5	8
0	0	-1	-3/5	0	3/10	-1/10	1/2

Optimal solution is $x_1^* = 8, x_2^* = 1/2, x_3^* = 0, z^* = 39/2$

g.

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$c_B B^{-1} N - c_N = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} -7 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 7 & -3 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 7 & 2 \end{bmatrix}$$

(4)

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
-1	7	2	5	4	0	0	0	0
0	-2	-4	-7	-1	1	0	0	-5
0	-8	-4	-6	-4	0	1	0	-8
0	-3	-8	-1	-4	0	0	1	-4

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
-1	3	0	2	2	0	1/2	0	-4
0	6	0	-1	3	1	-1	0	3
0	2	1	3/2	1	0	1/4	0	2
0	13	0	11	4	0	-2	1	12

Optimal solution is $x_1^* = 0, x_2^* = 2, x_3^* = 0, x_4^* = 0, z^* = 4$

(5)

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
-1	5	3	5	0	4	0	0	0
0	1	0	1	0	0	1	0	5
0	0	-1	0	0	-1	0	1	-3
0	1	1	1	1	0	0	0	7

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
-1	5	0	5	0	1	0	3	-9
0	1	0	1	0	0	1	0	5
0	0	1	0	0	1	0	-1	3
0	1	0	1	1	-1	0	1	4

Optimal solution is $x_1^* = 0, x_2^* = 3, x_3^* = 0, x_4^* = 4, x_5^* = 0, z^* = 9$